Degree functions and projectively full ideals in
two-dimensional rational singularities that can be
desingularized by blowing up the unique maximal ideal

Veronique Van Lierde

Al Akhawayn University, Ifrane 53000, Morocco

Abstract
Let \((R, m)\) be a 2-dimensional rational singularity with algebraically closed
residue field and for which the associated graded ring is an integrally closed do-
main. According to Göhner, \((R, m)\) satisfies condition \((N)\): given a prime divi-
sor \(v\), there exists a unique complete \(m\)-primary ideal \(A_v\) in \(R\) with \(T(A_v) = \{v\}\)
and such that any complete \(m\)-primary ideal with unique Rees valuation \(v\), is
a power of \(A_v\). We use the theory of degree functions developed by Rees and
Sharp as well as some results about regular local rings, to investigate the degree
coefficients \(d(A_v, v)\). As an immediate corollary, we find that for a simple
complete \(m_1\)-primary ideal \(I_1\) in an immediate quadratic transform \((R_1, m_1)\) of
\((R, m)\), the inverse transform of \(I_1\) in \(R\) is projectively full.

Key words: degree function, Rees valuation, quadratic transformation,
2-dimensional rational singularity, projectively full ideal

2000 MSC: 13B22, 13H10

1. Introduction

The purpose of this paper is to study the degree coefficients \(d(A_v, v)\) of
one-fibered ideals \(A_v\) in a 2-dimensional rational singularity \((R, m)\) with alge-
braically closed residue field and for which the associated graded ring is an inte-
grally closed domain. In [6, prop. 3.5] it was shown that, for every prime
divisor \(v\) of a 2-dimensional regular local ring \((R, m)\) with algebraically closed
residue field, there exists a unique complete \(m\)-primary ideal \(I\) of \(R\) such that
\(T(I) = \{v\}\) and \(d(I, v) = 1\). It was also shown that this is no longer true if \((R, m)\)
is not regular [6, example 4.1]. In this paper we work in 2-dimensional rational
singularities that can be desingularized by blowing up the unique maximal ideal.
We will now give a brief overview of the definition and of some properties of a
2-dimensional rational singularity. Let \((R, m)\) be a 2-dimensional analytically

Email address: v.lierde@aui.ma (Veronique Van Lierde)
normal local domain with infinite residue field. If \( I \) is an \( m \)-primary ideal of \( R \), then
\[
\ell \left( \frac{R}{I^n} \right) = \tau_0(I) \left( \frac{n + 1}{2} \right) - \tau_1(I) \left( \frac{n}{1} \right) + \tau_2(I)
\]
for all \( n > 0 \), where the coefficients \( \tau_i(I) \) are integers. The function \( H_I(n) := \ell \left( \frac{R}{I^n} \right) \) is called the normal Hilbert function of \( I \) and the polynomial \( P_I(n) := \tau_0(I) \left( \frac{n + 1}{2} \right) - \tau_1(I) \left( \frac{n}{1} \right) + \tau_2(I) \) is called the normal Hilbert polynomial of \( I \). A 2-dimensional analytically normal local ring \((R, m)\) (with infinite residue field) is called a rational singularity if \( \tau_2(I) = 0 \) for any \( m \)-primary ideal \( I \) of \( R \). If one replaces the condition analytically normal in the previous definition by normal and analytically unramified, then \((R, m)\) is called pseudo-rational [12].

For a 2-dimensional analytically normal local ring \((R, m)\) with infinite residue field, the following are equivalent:

- \((R, m)\) is a rational singularity.
- For any \( m \)-primary ideal \( I \) of \( R \) one has \( H_I(n) = P_I(n) \) for all \( n \geq 0 \).
- For any \( m \)-primary ideal \( I \) of \( R \) and any minimal reduction \((x, y)\) of \( I \) one has \((x, y)^{\frac{n}{1}} = I^{n + 1} \) for all \( n \geq 1 \).
- For any complete \( m \)-primary ideal \( I \) of \( R \) one has \( e(I) = \ell \left( \frac{R}{I^n} \right) - 2\ell \left( \frac{R}{I} \right) \) and \( I^n = I^{n+1} \) for all \( n \geq 1 \).

In a 2-dimensional rational singularity \((R, m)\), the product of complete ideals is complete again. S.D. Cutkosky has shown in [4] that the converse also holds if \((R, m)\) is a 2-dimensional analytically normal local domain with algebraically closed residue field. A 2-dimensional rational singularity has minimal multiplicity, i.e.
\[
\text{emb dim} R = e(R) + 1.
\]

According to Göhner [8, corollary 3.11], a 2-dimensional rational singularity satisfies condition \((N)\). Given a prime divisor \( v \), there exists a unique complete \( m \)-primary ideal \( A_v \) in \( R \) with \( T(A_v) = \{v\} \) and such that any complete \( m \)-primary ideal with unique Rees valuation \( v \), is a power of \( A_v \). Also, there exists a positive integer \( s \) such that for every complete \( m \)-primary ideal \( I \) in \( R \), there is a unique decomposition \( I^s = \prod_{v \in T(I)} A_v^s \). Here \( T(I) \) denotes the set of all Rees valuations of \( I \).

Using (1) and the work of Göhner [8, section 2], we study, for a rational singularity, the behavior of the degree coefficients \( d(I, v) \) under a quadratic transformation. These coefficients were introduced by D. Rees in [11]. Let \((R, m)\) be a local domain with quotient field \( K \). With an \( m \)-primary ideal \( I \) of \( R \), Rees associated an integer-valued function \( d_I \) on \( m \setminus \{0\} \) as follows:
\[
d_I(x) = e \left( \frac{I + xR}{xR} \right)
\]
where $e(I, v)$ is the multiplicity of $I$ at $v$. For every prime divisor $v$ of $R$, there is an associated non-negative integer $d(I, v)$, with $d(I, v) = 0$ for all except finitely many $v$, such that
\[
d(I, v) = \sum_v d(I, v)v(x) \quad \forall 0 \neq x \in m
\]
where the sum is over all prime divisors $v$ of $R$ ([11], Thm. 3.2). By a prime divisor $v$ of $R$ we mean a discrete valuation $v$ of $K$ which is non-negative on $R$ and has center $m$ on $R$ and whose residual transcendence degree is $\dim R - 1$. The set of all prime divisors of $R$ will be denoted by $P(R)$. In case $(R, m)$ is analytically unramified, $d(I, v) \neq 0$ for all $v \in P(R)$ that are Rees valuations of $I$ as defined by Rees in [11], whereas $d(I, v') = 0$ for all other prime divisors $v'$ of $R$. We will give more background information on degree functions and on quadratic transformations in section 2.

We will assume for the remainder of this section that $(R, m)$ is a 2-dimensional rational singularity with algebraically closed residue field $R/m$. We will also assume that the associated graded ring $gr_m R$ is an integrally closed domain. This implies that $\text{ord}_R$ is a valuation and that $B\ell_m R$ is a desingularization of $R$ [8, 9]. Here $B\ell_m R$ denotes the scheme $\text{Proj}(\oplus_{n \geq 0} m^n)$ obtained by blowing up $m$. Let $I$ be a complete $m$-primary ideal of $(R, m)$ and let $R_1, \ldots, R_n$ be the immediate base points of $I$. In [7, corollary 3.4] the following inequality for the multiplicity $e(I)$ of $I$ was obtained:
\[
e(I) \leq e(m)\text{ord}_R(I)^2 + e(I^{R_1}) + \ldots + e(I^{R_n})
\]
where $I^{R_i}$ denotes the transform of $I$ in $R_i$. The rings $R_1, \ldots, R_n$ are 2-dimensional regular local rings.

In section 3 we obtain the following result. Let $v \neq \text{ord}_R$ be a prime divisor of $R$ and let $A_v$ be the unique complete $m$-primary ideal of $R$ with $T(A_v) = \{v\}$ and such that any complete $m$-primary ideal of $R$ with $v$ as its unique Rees valuation, is a power of $A_v$. Let $R_1$ denote the unique immediate base point of $A_v$ and let $A_v^{R_i}$ denote the transform of $A_v$ in $R_i$. In theorem 3.3 we will show that
\[
d(A_v, v) = d(A_v^{R_i}, v).
\]
For a simple complete $m_1$-primary ideal $I_1$ in an immediate quadratic transform $(R_1, m_1)$ of $(R, m)$ with $T(I_1) = \{v\}$, we find that the inverse transform $I$ of $I_1$ in $R$ is a complete $m$-primary ideal of $R$ with $T(I) = \{v\}$ or $T(I) = \{v, \text{ord}_R\}$ and that $d(I, v) = 1$. We also give an example.

In section 4, we apply these results to provide a short proof for the fact that the inverse transform in $R$ of a simple complete $m_1$-primary ideal of an immediate quadratic transform $(R_1, m_1)$, is projectively full.

2. Background

Let $(R, m)$ be a 2-dimensional Noetherian local domain with fraction field $K$. The integral closure of an ideal $I$ of $R$ is denoted by $\overline{I}$. The ideal $I$ is called
integrally closed or complete if $\bar{I} = I$.

We will now briefly recall the definition of the Rees valuations and the Rees valuation rings of an $m$-primary ideal $I$ of $R$. Let $t$ be an indeterminate over $R$ and let $R[t, t^{-1}]$ be the following subring of $R\langle t, t^{-1} \rangle$: $R[t, t^{-1}] = \oplus_{n\in\mathbb{Z}} I^n t^n$ where $I^n = R$ if $n \leq 0$. Let $\bar{R}[t, t^{-1}]$ denote the integral closure of $R[t, t^{-1}]$ in its fraction field $K(t)$ and let $\{P_1, \ldots, P_n\}$ be the set of minimal primes of $(t^{-1})\bar{R}[t, t^{-1}]$. Then $\bar{R}[t, t^{-1}]$ is a Krull domain and each $P_i$ is a height one prime and consequently, $(\bar{R}[t, t^{-1}])_{P_i}$ is a discrete valuation ring of $K(t)$ for $i = 1, \ldots, n$. The Rees valuation rings of $I$ are

$$V_i := (\bar{R}[t, t^{-1}])_{P_i} \cap K \quad i = 1, \ldots, n.$$  

The corresponding discrete valuations $v_1, \ldots, v_n$ are called the Rees valuations of $I$ and the set of these Rees valuations is denoted by $T(I)$:

$$T(I) = \{v_1, \ldots, v_n\}.$$  

Using the Rees valuation rings of $I$, the integral closure $\bar{I}$ of $I$ is given by

$$\bar{I} = \cap_{i=1}^n I V_i \cap R.$$  

As written in the introduction, the degree function $d_I$ of an $m$-primary ideal $I$ in a Noetherian local domain $(R, m)$ can be written as follows:

$$d_I(x) = \sum_{v \in \mathcal{P}(R)} d(I, v)v(x) \quad \forall 0 \neq x \in m.$$  

In [14] Rees and Sharp have proved that the integers $d(I, v)$ are uniquely determined by the previous condition, i.e. suppose that

$$\sum_{v \in \mathcal{P}(R)} d(I, v)v(x) = \sum_{v \in \mathcal{P}(R)} d'(I, v)v(x) \quad \forall 0 \neq x \in m$$  

then $d(I, v) = d'(I, v)$ for every prime divisor $v$ of $R$. From this uniqueness it follows that for $m$-primary ideals $I$ and $J$ in a 2-dimensional Noetherian local domain $(R, m)$, one has that

$$d(IJ, v) = d(I, v) + d(J, v)$$  

for every prime divisor $v$ of $R$ [14, lemma 5.1]. If we make the additional assumption that $R$ is analytically unramified and normal, then this implies that

$$T(IJ) = T(I) \cup T(J).$$  

In [14, theorem 4.3] Rees and Sharp have shown that for an $m$-primary ideal $I$ in a 2-dimensional local domain $(R, m)$, the multiplicity $e(I)$ of $I$ is given by

$$e(I) = \sum_{v \in \mathcal{P}(R)} d(I, v)v(I).$$  

4
For $I$ and $J$ $m$-primary ideals in a 2-dimensional Cohen-Macaulay local domain $(R, m)$, Rees and Sharp define

$$d_I(J) = \min\{d_I(x) \mid 0 \neq x \in J\}$$

and they have proved [14, theorem 5.2] that

$$d_I(J) = \sum_{v \in P(R)} d(I, v) v(J)$$

and

$$d_I(J) = d_J(I) = e_1(I|J)$$

Here $e_1(I|J)$ denotes the mixed multiplicity of $I$ and $J$ and $e_1(I|J)$ is defined by $e(I.J) = e(I) + 2e_1(I|J) + e(J)$ [15, p. 1037].

We end this section with the following result of Rees and Sharp [14, corollary 5.3]. Let $I$ and $J$ be $m$-primary ideals in the 2-dimensional Cohen-Macaulay local domain $(R, m)$.

Then the following three statements are equivalent:

1. $\bar{I} = \bar{J}$
2. $d_I(x) = d_J(x)$ $\forall x \in m \setminus \{0\}$
3. $d(I, v) = d(J, v)$ $\forall v \in P(R)$

Finally, we briefly recall the following notions: immediate quadratic transform $R_1$ of $R$, transform $I_1$ of an ideal $I$ of $R$ in $R_1$, immediate base point $R_1$ of an ideal $I$ of $R$. From now till the end of the introduction we shall assume that the local ring $(R, m)$ is a 2-dimensional rational singularity with infinite residue field and for which the associated graded ring is an integrally closed domain. If $x \in m \setminus m^2$ and if $N$ is a maximal ideal in $R[\frac{m}{x}]$ lying over $m$ (i.e. $N \cap R = m$), then the ring

$$R_1 = R[\frac{m}{x}]_N$$

is called an immediate (or a first) quadratic transform of $R$. Let $I$ be an $m$-primary ideal of $R$. If $\text{ord}_R(I) = r$ (i.e. $I \subseteq m^r$ but $I \not\subseteq m^{r+1}$), then we have in $R_1$ that

$$IR_1 = x^rI_1$$

where $I_1$ denotes an ideal in $R_1$ called the transform of $I$ in $R_1$. In case $I_1 \neq R_1$, we say that $(R_1, m_1)$ is an immediate base point of $I$. Here $m_1$ denotes the maximal ideal of the local ring $R_1$. A given $m$-primary ideal $I$ in $R$ has only finitely many immediate base points.

3. Main result

Let $(R, m)$ be a 2-dimensional rational singularity with algebraically closed residue field $R/m$, and for which the associated graded ring $gr_mR$ is an integrally closed domain. This implies that $T(m) = \{\text{ord}_R\}$ and that $B\ell_mR$ is a desingularization of $R$. 

5
Lemma 3.1. Let \((R, m)\) be a 2-dimensional rational singularity with algebraically closed residue field, and suppose that the associated graded ring is an integrally closed domain. Let \(I\) be a complete \(m\)-primary ideal in \(R\) with \(T(I) = \{v\}\) and \(v \neq \text{ord}_R\). Then \(I\) has only one immediate base point \(R_1\). Let \(I_1\) denote the transform of \(I\) in \(R_1\). Then \(T(I_1) = \{v\}\) and \(d(I, v) \leq d(I_1, v)\).

Proof. Let \((R_1, m_1)\) be the unique local ring of the complete normal model \(B\ell_m R\) dominated by the valuation ring \((V, m_V)\) of \(v\). Since \((V, m_V)\) is the unique Rees valuation ring of \(I\), it follows that \((R_1, m_1)\) is the unique immediate base point of \(I\). Because of [8, proposition 2.9], there is an integer \(e > 0\) such that the transform of \(I'\) in \(R_1\) has \(v\) as its unique Rees valuation. Since \(R_1\) is regular and hence a UFD, we can take \(e = 1\) and so \(T(I_1) = \{v\}\).

Let \(R_1\) be of the following form: \(R_1 = R[[\frac{m}{m}]]_N\), with \(x \in m \setminus m^2\) and \(N\) a maximal ideal in \(R[[\frac{m}{m}]]\) lying over \(m\). Let \(r := \text{ord}_R(I)\). Then \(IR_1 = x^rI_1\).

From section 2, it follows that \(e(I) = d(I, v)\). Since \(I\) has one immediate base point \(R_1\), it follows from (1) that

\[ e(I) \leq e(m)r^2 + e(I_1). \]

Since \(e(m) = d(m, \text{ord}_R)\) and since \(v(I) = rv(m) + v(I_1)\), this implies that

\[ d(I, v)(rv(m) + v(I_1)) \leq d(m, \text{ord}_R)r^2 + d(I_1, v)v(I_1). \]

From the theory of degree functions, it follows that \(d(I, v)v(m) = d_I(m) = d_m(I) = d(m, \text{ord}_R)r\). So

\[ d(I, v)v(I_1) \leq d(I_1, v)v(I_1). \]

Since \(v(I_1) > 0\), it follows that \(d(I, v) \leq d(I_1, v)\). \(\Box\)

Let \(v \neq \text{ord}_R\) be a prime divisor of \(R\) and let \(A_v\) be the unique complete \(m\)-primary ideal of \(R\) with \(T(A_v) = \{v\}\) and such that any complete \(m\)-primary ideal of \(R\) with \(v\) as its unique Rees valuation, is a power of \(A_v\). Let \(R_1\) denote the unique immediate base point of \(A_v\) and let \(A_v^{R_1}\) denote the transform of \(A_v\) in \(R_1\). Then we will show that

\[ d(A_v, v) = d(A_v^{R_1}, v). \]

In order to do this, we need the notion of inverse transform. Let \(I_1\) be a complete \(m_1\)-primary ideal in an immediate quadratic transform \((R_1, m_1)\) of \((R, m)\). Then \(R_1\) is of the form \(R_1 = R[[\frac{m}{m}]]_N\), with \(x \in m \setminus m^2\) and \(N\) a maximal ideal in \(R[[\frac{m}{m}]]\) lying over \(m\). Let \(a\) be the smallest positive integer so that \(a^I_1\) is extended from \(R\) i.e. there exists an ideal \(J\) of \(R\) such that \(a^I_1 = JR_1\). Then

\[ I := a^I_1 \cap R \]

is called the inverse transform of \(I_1\) in \(R\). It is clear that \(a^I_1 = IR_1\) and \(IR_1 \cap R = I\), so \(I\) is contracted from \(R_1\). Note also that \(a = \text{ord}_R(I)\). Since \(I_1\) is \(NR[[\frac{m}{m}]]_N\)-primary, there is exactly one \(N\)-primary ideal in \(R[[\frac{m}{m}]]\), say \(I'\), such that \(I_1 = I'_N\).

The following result is known but we include a proof for clarity.
Let $I'$ be the transform of $I$ in $R[\frac{m}{x}]$, i.e., $IR[\frac{m}{x}] = x^a I'$. The inverse transform $I$ of $I_1$ has $R_1$ as its unique immediate base point.

Proof. As the residue field of $R$ is algebraically closed, we may suppose without loss of generality that the element $x \in m \setminus m^2$ is chosen in such a way that all immediate base points of $I$ are localizations of $R[\frac{m}{x}]$. Let $b$ be the smallest positive integer such that $x^bI'$ is extended from $R$, i.e., $x^bI' = (x^bI' \cap R)R[\frac{m}{x}]$.

This implies that $x^bI_1$ is extended from $R$ and hence $b \geq a$. Since $I = x^aI' \cap R$, it is sufficient to prove that $b = a$. Suppose $b > a$, then

$$(x^bI' \cap R)R[\frac{m}{x}]|_N = m^{b-a} \cdot IR[\frac{m}{x}]|_N$$

and since $x^bI' \cap R$ and $m^{b-a}I$ are contracted from $R[\frac{m}{x}]|_N$, contraction to $R$ implies

$$x^bI' \cap R = m^{b-a}I.$$ 

Extension to $R[\frac{m}{x}]$ yields

$$x^bI' = x^{b-a}IR[\frac{m}{x}]$$

and this implies that

$$x^aI' = IR[\frac{m}{x}]_1.$$ 

Thus $x^aI'$ is extended from $R$, so by the choice of $b$ one has $a \geq b$, which contradicts with $b > a$.

Since $N$ is the only height two prime ideal containing $I'$ and since $IR[\frac{m}{x}] = x^aI'$, it follows that $R_1$ is the unique immediate base point of $I$. □

We are now ready to prove our main result.

Theorem 3.3. Let $(R,m)$ be a 2-dimensional rational singularity with algebraically closed residue field, and suppose that the associated graded ring is an integrally closed domain. Let $v \neq \text{ord}_R$ be a prime divisor of $R$ and let $A_v$ be the unique complete $m$-primary ideal of $R$ with $T(A_v) = \{v\}$ and such that any complete $m$-primary ideal of $R$ with $v$ as its unique Rees valuation, is a power of $A_v$. Let $R_1$ denote the unique immediate base point of $A_v$ and let $A_v^{R_1}$ denote the transform of $A_v$ in $R_1$. Then $d(A_v, v) = d(A_v^{R_1}, v)$.

Proof. Since $A_v$ is one-fibered, it follows from lemma 3.1 that $A_v$ has one immediate base point $(R_1, m_1)$ and that $T(A_v^{R_1}) = \{v\}$. Then $R_1$ is of the form $R_1 = R[\frac{m^2}{x}]$, with $x \in m \setminus m^2$ and $N$ a maximal ideal in $R[\frac{m^2}{x}]$ lying over $m$. Since $A_v^{R_1}$ is a complete $m_1$-primary ideal in $R_1$ with unique Rees valuation $v$, and since $(R_1, m_1)$ is a 2-dimensional regular local ring, it follows from Zariski’s Unique Factorization Theorem [16] and from [6, proposition 3.5] that $A_v^{R_1} = I_1^h$, with $I_1$ a simple complete $m_1$-primary ideal in $R_1$, $T(I_1) = \{v\}$ and $d(I_1, v) = 1$.

Since $T(A_v) = \{v\}$, it follows from lemma 3.1 that $d(A_v, v) \leq d(I_1^h, v)$. Since $d(I_1, v) = 1$, we have $d(I_1^h, v) = k$.

Let $I$ be the inverse transform of $I_1$ in $R$. Then $x^aI_1 = IR_1$ and $a = \text{ord}_R(I)$. Since $I = x^aI_1 \cap R$, it follows that $I$ is a complete $m$-primary ideal of $R$.
Because of lemma 3.2, $R_1$ is the only immediate base point of $I$. This implies that the blow-up $Bl_{I_1} R$ of $R$ at $I_1$ is obtained from $Bl_{m} R$ by blowing up $R_1$ at $I_1$ while leaving all the other local rings of $Bl_{m} R$ unaltered.

It follows that

$$T(Im) = T(I) \cup T(m) = \{v, \ord_R\}.$$  

This implies that $T(I) = \{v\}$ or $T(I) = \{v, \ord_R\}$.

Case I. $T(I) = \{v\}$.

Since $I^{R_1} = I_1$, lemma 3.1 implies that $d(I, v) \leq d(I_1, v) = 1$. Since $I$ is a complete $m$-primary ideal of $R$ with $T(I) = \{v\}$, it follows that $I = A_v^s$ for some positive integer $e$. So

$$1 = d(I, v) = ed(A_v, v).$$

So in this case, $e = 1$ and $I = A_v$, and $d(A_v, v) = 1$.

Case II. $T(I) = \{v, \ord_R\}$.

In this case, it follows from [8, corollary 3.11] that

$$I^s = A_v^s m^f$$

for some positive integers $s, e, f$. Since $I$ has one immediate base point $R_1$ and since $(I^s)^{R_1} = (I^{R_1})^s$, it follows that $I^s$ has $R_1$ as unique immediate base point.

We now compute $(I^s)^{R_1}$ in two ways. On the one hand, $(I^s)^{R_1} = (I^{R_1})^s = I_1^s$. On the other hand, $(I^s)^{R_1} = (A_v^s m^f)^{R_1} = (A_v^s)^{R_1} = (A_v^{R_1})^e = I_1^{ke}$. So it follows that

$$I_1^s = I_1^{ke}.$$  

This implies that $s = ke$, so $I^s = I_1^{ke} = A_v^s m^f$. In combination with lemma 3.1 and with the fact that $A_v^{R_1} = I_1^k$, this implies that

$$ked(I, v) = ed(A_v, v) \leq ed(A_v^{R_1}, v) = ed(I_1, v) = ek.$$  

So $d(I, v) = 1$ and $ke = ed(A_v, v) = ed(A_v^{R_1}, v)$. Consequently, $d(A_v, v) = k$ and $d(A_v, v) = d(A_v^{R_1}, v)$. □

The following result is contained in the proof of theorem 3.3. We formulate it separately since it gives a converse of lemma 3.1.

**Corollary 3.4.** Let $(R, m)$ be a 2-dimensional rational singularity with algebraically closed residue field, and suppose that the associated graded ring $gr_m R$ is an integrally closed domain. Let $(R_1, m_1)$ be an immediate quadratic transformation of $(R, m)$ and let $I_1$ be a simple complete $m_1$-primary ideal of $(R_1, m_1)$ with unique Rees valuation $v$. Let $I$ be the inverse transform of $I_1$ in $R$. Then $I$ is a complete $m$-primary ideal of $R$ with $T(I) = \{v\}$ or $T(I) = \{v, \ord_R\}$, and $d(I, v) = 1$.

Note that in the special case where $(R, m)$ is a 2-dimensional regular local ring, it follows that $T(I) = \{v\}$ in coll. 3.4, since one can use [16, proposition 3]
in that case (see the proof of prop. 3.4 in [6]). The following example illustrates the case \( T(I) = \{ v, \text{ord}_R \} \).

**Example 3.5** Consider the following 2-dimensional rational singularity:

\[
R := \frac{k[X, Y, Z]|_{(X, Y, Z)}}{(X^2Y - Y^2 - Z^2 - XZ)|_{(X, Y, Z)}}
\]

with \( k \) an algebraically closed field. Let \( x, y, z \) denote the images of \( X, Y, Z \) in \( R \). Then \( R = k[x, y, z]|_{(x, y, z)} \) with \( x^2y = y^2 + z^2 + xz \) and \( m = (x, y, z) \) is the unique maximal ideal of \( R \). Consider

\[
R_1 := R[\frac{m}{x}]_N \quad N = (x, \frac{y}{x}, \frac{z}{x}).
\]

Let \( I \) be the following \( m \)-primary ideal in \( R \):

\[
I = (x^2, y, z).
\]

It follows from [5, lemma 1] that \( I \) is the inverse transform of the unique maximal ideal \( m_1 \) of \( R_1 \) and that \( T(I) = \{ v, \text{ord}_R \} \), where \( v \) is the \( \text{ord}_{R_1} \)-valuation. Also, \( R_1 \) is the unique immediate base point of \( I \).

According to coll. 3.4, we have \( d(I, v) = 1 \). In this particular example, we can verify this as follows. Note that \( d_m(I) = d_I(m) \), so

\[
d_m(\text{ord}_R)\text{ord}_R(I) = d(I, \text{ord}_R)\text{ord}_R(m) + d(I, v)\text{ord}(m).
\]

Since \( d(m, \text{ord}_R) = 2 \) and \( \text{ord}_R(I) = 1 \), and since \( d(I, \text{ord}_R) \) and \( d(I, v) \) are positive integers, it follows that

\[
d(I, v) = d(I, \text{ord}_R) = 1.
\]

### 4. Projectively full ideals

In this section we use the preceding results to give a class of projectively full ideals. The notion of projectively full ideals was introduced by Ciuperca, Heinzer, Ratliff Jr. and Rush in [1]. A regular ideal \( I \) in a Noetherian ring \( R \) is called projectively full if \( \overline{I} \) (\( n \in \mathbb{N}_0 \)) are the only integrally closed ideals that are projectively equivalent to \( I \). Two ideals \( I \) and \( J \) are called projectively equivalent if there exist positive integers \( m \) and \( n \) such that \( \overline{I} = \overline{I} \). For a regular ideal \( I \) of \( R \), let \( \mathbf{P}(I) \) denote the set of all integrally closed ideals that are projectively equivalent to \( I \). Then \( \mathbf{P}(I) \) is called projectively full if there exists a projectively full ideal \( J \) that is projectively equivalent to \( I \). Projectively equivalent ideals were introduced by Samuel in [13]. The theory of projectively equivalent ideals was further developed by McAdam, Ratliff Jr. and Sally in [10] and by Ciuperca, Heinzer, Ratliff Jr. and Rush in [1, 2]. Lipman proved that \( \mathbf{P}(I) \) is projectively full for each complete \( m \)-primary ideal \( I \) if \( (R, m) \) is a 2-dimensional normal local domain that has a rational singularity. A proof
can be found in [3, section 6]. In [5, thm. 1] Debremaeker has shown that first neighborhood complete ideals in 2-dimensional Muhly local domains \((R, m)\) are projectively full, by giving an explicit description of these ideals. A first neighborhood complete ideal in \((R, m)\) is the inverse transform of the maximal ideal of an immediate quadratic transformation of \((R, m)\). For a 2-dimensional Muhly local domain that is a rational singularity, we can generalize this result to the inverse transform of any simple complete ideal. Moreover, we are able to give a very short and simple proof using coll. 3.4.

**Theorem 4.1.** Let \((R, m)\) be a 2-dimensional rational singularity with algebraically closed residue field, and suppose that the associated graded ring \(gr_mR\) is an integrally closed domain. Let \((R_1, m_1)\) be an immediate quadratic transformation of \((R, m)\) and let \(I_1\) be a simple complete \(m_1\)-primary ideal of \((R_1, m_1)\) with unique Rees valuation \(v\). Then the inverse transform \(I\) of \(I_1\) in \(R\) is projectively full.

**Proof.** From coll. 3.4 it follows that \(T(I) = \{v\}\) or \(T(I) = \{v, \text{ord}_R\}\). In case \(T(I) = \{v\}\), we have \(I = A_v\) and \(A_v\) is projectively full. So we can assume that \(T(I) = \{v, \text{ord}_R\}\). Let \(J\) be a complete \(m\)-primary ideal of \(R\) that is projectively equivalent to \(I\). Then there exist positive integers \(m\) and \(n\) such that

\[ J^n = J^m = I^m. \]

This implies that \(T(J) = \{v, \text{ord}_R\}\) and

\[ d(J^n, v) = nd(J, v) = md(I, v) = m. \]

So \(d(J, v) = \frac{m}{n}\) and \(\frac{m}{n}\) is a positive integer. Also,

\[ d(J^n, \text{ord}_R) = d(I^m, \text{ord}_R) = md(I, \text{ord}_R). \]

Since \(d(J, v) = \frac{m}{n}d(I, v)\), \(d(J, \text{ord}_R) = \frac{m}{n}d(I, \text{ord}_R)\) and \(T(J) = T(I^\frac{m}{n})\), it follows from the theory of degree functions that \(J = I^\frac{m}{n}\). \(\square\)

**Acknowledgement**

The author would like to thank the referee for his or her helpful remarks.

**References**


