Computing Axis of Symmetry Using Constrained High Curvature Points Matching

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Abstract

Finding the axis of symmetry of two curves is posed as a correspondence problem between points of significant curvature along these curves. First, a novel and simple algorithm for computing these points is developed. Its main strength is that it does not use curvature values, which are sensitive to noise, but uses a robust indication of them. The detected points are then fed to another algorithm which integrates the computation of the parameters of the axis of symmetry (ρ, θ) *into the correspondence algorithm. This algorithm is simple and intuitive. It uses natural ordering of points of significant curvature along curves, the length* ρ*, and the angle* θ *(of the normal vector which connects the axis to the origin), to constrain the search for the true matches. The parameters of the axis* (ρ, θ) *are then evaluated from the positions of the longest match of pairs of points of significant curvature.This technique is robust and copes particularly well with noise and occlusion problems at curve ends as shown by the experimental results.*

Keywords: *Axis of symmetry, Correspondence, Matching, Curvature, High Curvature Points, Curves.*

1 Introduction

A very widely used characteristic of objects are their symmetry sets. They have been used in image description, object recognition, stereoscopic matching, viewpoint-invariant representations [11], and as shape descriptors in model based recognition [5], *et cetera.* Axis of symmetry, in particular, have been used to segment MR images of the brain into the left and right hemispheres [10]. They are also used in Robotics for picking up objects.

In this paper we present a robust algorithm for computing axis of symmetry of an object using points of significant curvature. We first present an algorithm that yields points of significant curvature.

2 Previous techniques

Previous techniques can be divided into two catgories: those that use segmented images and those that don't. In [1] a voting technique is applied to non-segmented images to determine axes of skewed symmetries using local skewed symmetries. This technique suffers, however, from the number of steps necessary to yield the parameters of axes. More importantly, it suffers from the use of curvature values which are known to be sensitive to noise. In [8], moments of objects in segmented images are used to compute the axis of symmetry. Relative invariants computed from bitangents in objects are used to determine symmetry in [9]. One way of extracting symmetry sets, in general, and axes of symmetry, in particular, of segmented images is presented in [3], it consists of computing the locus of centers of circles bitangent to a plane curve. Our technique uses segmented images to determine axis of symmetry of objects.

3 Estimating curvature

Points of maximum curvature are commonly believed to be the most perceptually significant points on digital curves, and as such have been used as shape features in both 2-D and 3-D object recognition by many researchers.

In real Euclidean plane, curvature is defined as the rate of change of slope as function of an arc length. For the curve $y = f(x)$ this may be expressed as:

$$\frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}} \qquad (1)$$

For the digital case it is not clear how to define an equivalent measure of slope as it includes computing second derivatives, and thus is very sensitive to noise. A wide variety of point curvature estimators have been proposed such as curvature estimation from cubic fitting [2], and curvature estimation using smoothed k-cosines [6]. Evaluation of point curvature estimators was presented in [4]. These methods require a lot of computations as they try to estimate the exact value of the curvature. The method presented here gives an indication of the curvature without really computing the curvature. It yields the locations of points of significant curvature based on ratio of distances between two points A and B (arc distance $|\widehat{AB}|$ and Euclidean distance $|AB|$). It requires no computations of second derivatives and hence is very simple and efficient.

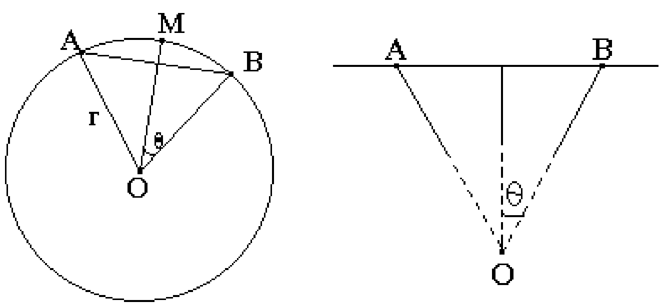

Figure 1. *M is the point under focus. The larger the ratio $\frac{|\widehat{AB}|}{|AB|}$ the higher the curvature at point M. For the line (AB) O goes to infinity and thus θ goes to* 0

To illustrate our idea, consider a cercle $C(O,r)$ (see **Figure** 1).

We have: $|\widehat{AB}| = 2\,\theta\, r$ and $|AB| = 2\, r\, sin\theta$. Hence,

$$\frac{|\widehat{AB}|}{|AB|} = \frac{\theta}{sin\theta} \qquad (2)$$

Thus, given a θ, all points on a cercle have the same curvature significance.

Consider a line (AB) (see **Figure** 1). θ goes to 0 and the ratio in **(2)** goes to one. Thus there is no point of significant curvature on a line.

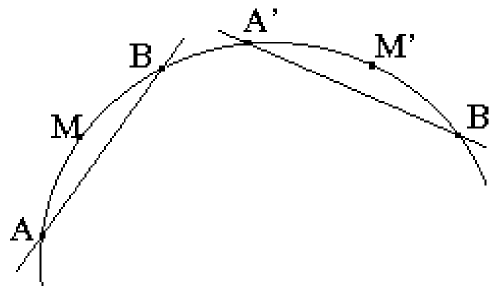

Figure 2. *M is the point under focus. The larger the ratio $\frac{|\widehat{AB}|}{|AB|}$ the higher the curvature at point M.*

Consider points A, B, A' and B' on a curve C (see **Figure** 2). Let $|\widehat{AB}|$ be the length of the arc $\widehat{AB}$ and $|AB|$ the Euclidean distance between A and B. Let M be the mid-point of $\widehat{AB}$ on the arc. We can have a good estimation of the curvature at point M by considering the ratio $\frac{|\widehat{AB}|}{|AB|}$. Thus, the larger this ratio, the higher the curvature at point M.

Formally, the algorithm is as follows:

1. For each pair of points (A,B) on the curve such that $|AB| = \mathbf{d}$,
2. If $\frac{|\widehat{AB}|}{|AB|} > threshold$ declare the mid-point (M) on the arc $\widehat{AB}$, point of high curvature.

d is a global parameter of the algorithm. It is the euclidean distance between the two points under focus A and B.

This method has the advantage of being very simple and easy to implement.

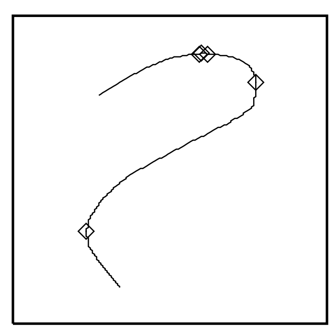

Figure 3. *Result of the algorithm on an artificial curve. The diamons represent the points of significant curvature detected by the algorithm.*

4 Finding axis of symmetry

A symmetry axis is a line whose equation in polar form is:

$$y\,cos\theta - x\,sin\theta + \rho = 0 \tag{3}$$

Our goal is to find ρ and θ parameters of the axis of symmetry for two given curves.
Let $(P_j, P'_j), (P_k, P'_k)$ two pairs of corresponding points on curves C and C' (see **Figure** 4).

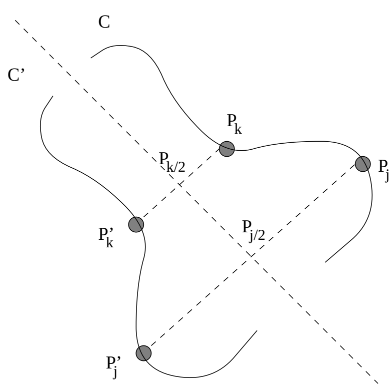

Figure 4. *Parameters $\rho_{j,k}$ and $\theta_{j,k}$ are computed from points $P_{j/2}$ and $P_{k/2}$ which are derived from corresponding Peaks (P_j, P'_j) and (P_k, P'_k).*

Let $P_{k/2} = (x_{k/2}, y_{k/2})^T$ and $P_{j/2} = (x_{j/2}, y_{j/2})^T$ be the respective midpoints of segments $[P_k, P'_k]$ and $[P_j, P'_j]$. Stating that the axis of symmetry goes through $P_{k/2}$ and $P_{j/2}$ gives the following parameters $\theta_{j,k}$ and $\rho_{j,k}$:

$$\theta_{j,k} = atan\left(\frac{a}{(b+c)}\right) \tag{4}$$

$$\rho_{j,k} = \frac{y_{k/2} - y_{j/2}}{2} cos\theta_{j,k} \quad - \quad \frac{x_{k/2} - x_{j/2}}{2} sin\theta_{j,k} \tag{5}$$

Where

$$a = \frac{(x_{k/2} - x_{j/2})(y_{k/2} - y_{j/2})}{2} \tag{6}$$

$$b = \frac{(x_{k/2} - x_{j/2})^2 - (y_{k/2} - y_{j/2})^2}{8} \tag{7}$$

$$c = \sqrt{a^2 + b^2} \tag{8}$$

A simple mathematical derivation shows that points $P_{k/2}$ and $P_{j/2}$ belong to the axis even if the symmetry is skewed.
The parameters $\rho_{j,k}$ and $\theta_{j,k}$ must be equal for every pair of corresponding points (P_j, P'_j) and $(P_k, P'k)$ because there is only one axis of symmetry. This is expressed as follows:

$$\forall j,k \quad \rho_{jk} = Cte = \rho \tag{9}$$

$$\theta_{jk} = Cte' = \theta \tag{10}$$

It follows then that, if correspondence between points of curves C and C' is established, the parameters ρ and θ of the axis of symmetry can be computed using equations (4)–(8) and any pair of corresponding points.

4.1 Moving the goal post

With this in mind, finding the axis of symmetry is posed as establishing correspondence between points of significant curvature. In computing correspondence, equations (9) and (10) play a fundamental role. These equations simply state that amongst all the possible pairings, we should retain the pairing $(P_1, P'_1), (P_2, P'_2) \cdots$ that satisfies the folowwing constraints:

$$\forall j,k \quad \forall h,l$$

$$[|\rho_{jk} - \rho_{hl}| \le \epsilon_\rho] \tag{11}$$

$$[|\theta_{jk} - \theta_{hl}| \le \epsilon_\theta] \tag{12}$$

where $\rho_{j,k}$ and $\theta_{j,k}$ are obtained from points $P_{j/2}$ and $P_{k/2}$ (see **Figure** 4) and ϵ_ρ (resp. ϵ_θ) is a predefined small threshold that guarantees the equality between ρ_{jk} and ρ_{hl} (resp. θ_{jk} and θ_{hl}).
Assuming that C's (resp. C''s) points of significant curvature P_i $i \in 1..n$ (resp. P'_i $i \in 1..n'$) have been ordered. The following subset of constraints is used in order to maintain efficiency:

$$\forall i \in \{1..n-2\},$$

$$[|\rho_{i,i+1} - \rho_{i+1,i+2}| \le \epsilon_\rho] \tag{13}$$

$$[|\theta_{i,i+1} - \theta_{i+1,i+2}| \le \epsilon_\theta] \tag{14}$$

where $\rho_{i,i+1}$ and $\theta_{i,i+1}$ are obtained from points $P_{i/2}$ and $P_{i+1/2}$ and $\rho_{i+1,i+2}$ and $\theta_{i+1,i+2}$ are obtained from points $P_{i+1/2}$ and $P_{i+2/2}$.
The technique of integrating the unknown parameters as constraints has been termed Constrained Peak Matching and has been successfully applied to computing the parameters of the 2D affine transform between two curves [7].
Furthermore, the matching between points of significant curvature must be one-to-one. This constraint stems from the physical nature of these points, i.e., a

point of high curvature in the scene always projects into a unique point in the image plane. Additionally, crossings are prohibited. This is dictated by the natural ordering on boundary points.

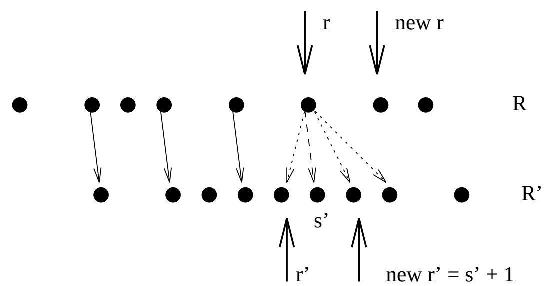

Figure 5. *Establishing correspondence between peaks. Peak r is currently under fucus. Among the M possible pairings (marked with dashed arrows), s' (marked with a dotted arrow) is the pairing that give a point m consistent with the other points m' (the middle of previously established pairs marked with solid arrows). The search continues with the peak marked new r under examination, and with $r' = s' + 1$.*

The final and desired pairing is a pairing between points of significant curvature of each curve such that the parameters of the axis of symmetry obtained from any subset of it lead to the same parameters ρ and θ.

4.2 Establishing correspondence

Let C and C' be the two curves, R and R' the arrays where maximum curvature points of C and C' are stored. At least two high curvature points have to be present in C and C'. Curves which do not satisfy this condition are ignored. To find the axis of symmetry of C and C' we have to find the longest match between points from R and R' and compute mid-points of the pairs yield by the matching. To find the longest symmetry set, an exhaustive search is employed. We proceed as follows: let $L = \{m\}$ be the list of mid-points (symmetry set) at a given stage. Initially L is empty. Let r and r' be the indices of the current curvature points of R and R' under focus (see **Figure** 5). For all peaks between r' and $r' + M$, select s' such that the middle M' of the new pair (r, s') is consistent with all existing points in L. If such peak exists, add M' to L. If $\#L > \#B$, (where B is the previously best recorded set of middle points, and $\#B$ its size), record L as the new best symmetry set, and continue with $r = r + 1$ and $r' = r' + 1$. If, however, no peak between r' and $r' + M$ can be constantly paired with r, continue with $r = r + 1$. M, the number of potential peaks from R' under scrunting each time, is a global parameter of the process.

The algorithm is as follows:

```
Search(R,N,r,R',N',r',L,l)

ARRAYS of PEAKS R,R'
LENGTHS N,N'
INDICES r,r'
LIST of PAIRS L
LENGTH l

BEGIN
    IF (r ≥ N OR r' ≥ N') THEN
        RETURN;

    FOR i=r' TO r'+M DO
        IF Consistent(R[r],R'[i],L,l) THEN
            L ← L ∪ {(r,i)} ;
            Search(R,N,r+1,R',N',i+1, L, l+1);
            IF Improvement(L,B) THEN
                B = L;
            END
        END
    END
END
```

Algorithm 4.1. *Algorithm for Constrained Peak Matching.*

Improvement (L,B) determines whether there is an improvement in the length of L with respect to the length of B. Clearly, if $\#L > \#B$ then there is an improvement, otherwise there is none. The function *Consistent(M,L,l)* returns 1 if M is consistent with the list of points in L (of length l) at the time of the call. Let M_{i-1} and M_i the two last points in L and M the point under focus. To check the consistency, we compute the parameters ρ_i and θ_i of the line $(M_{i-1}\, M_i)$ and ρ_{i+1} and θ_{i+1} of the line $(M_i\, M)$. If $|\rho_{i+1} - \rho_i| < \epsilon_\rho$ and $|\theta_{i+1} - \theta_i| < \epsilon_\theta$ then M is consistent with the points in L and *Consistent()* returns 1 otherwise, it returns 0. The search is launched with the following instructions:

```
B = ∅;
FOR i=0 TO M DO
    Search(R,N,i,R',N',0,∅,0);
END
```

The final ρ (resp. θ) for the symmetry axis are computed as the average of ρ_i (resp. θ_i) over all pairs of points in L (consecutive points):

$$\rho = \frac{\sum_{i=1}^{n} \rho_i}{n} \quad \text{and} \quad \theta = \frac{\sum_{i=1}^{n} \theta_i}{n}$$

Where $n = \#B$, and θ_i and ρ_i are the parameters of the line $(M_{i-1}\, M_i)$

5 Experiments and results

The number of pairs obtained is a good indication of the correctness of the computed parameters of the line. The ratio $r = \frac{\#B}{min(\#R,\#R')}$ is a self verifying indicator of the goodness of the model. The computed parameters of the axis of summetry are taken to be reliable if $r \geq 0.5$, which implies that 50% of the points of significant curvature are correctly matched in accordance with the constraints.
Typicaly, $M = 10$, $\epsilon_\rho = 1$, $\epsilon_\theta = 1$.

Curve	(a)	(b)
M	10	10
ϵ_θ	1.0	1.5
ϵ_ρ	1.0	1.5
$\#B$	14	59
$min(\#R, \#R')$	29	64
r	0.48	0.92

Table 1. *The parameters used to compute ρ and θ. The ratio r indicates the goodness of the matching (the higher, the better). See* **Figure** *6 for curves (a) and (b)*

Conclusion

The various experiments demonstrate the robustness of the approach presented in this paper to determine the parameters of the axis of symmetry of two curves. This robustness stems from the formulation of the parameters of the symmetry axis in terms of high curvature points pairings rather than curvature itself.

For further work, the authors are investigating the usage of this technique in computing other symmetry sets than axis.

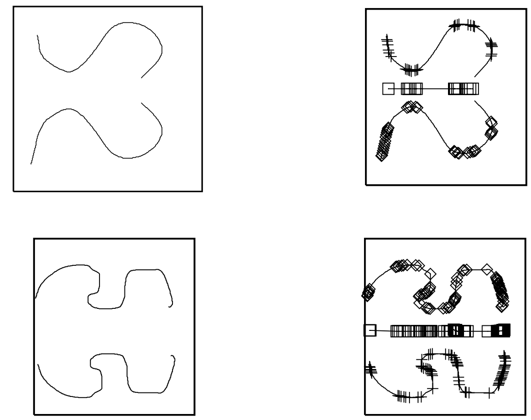

Figure 6. *(a) and (b) are a pair of symmetric artificial curves. The parameters used to compute the axis of symmetry are displayed in* **Table** *1. The points corresponding to the best match and the axis of symmetry are sketched on the figures on the right.*

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