Chapter 7: Rate of Return Analysis: Single Alternative

Sessions 16, 17
Dr Abdelaziz Berrado
Topics to Be Covered in Today’s Lecture

- Section 7.1: Interpretation of a Rate of Return Value
- Section 7.2: ROR using Present Worth
- Section 7.3: Cautions when using the ROR Method
Section 7.1: INTERPRETATION OF A RATE OF RETURN VALUE

- Rate of Return (ROR) also known by Internal Rate of Return (IRR), Return on Investment (ROI), and Profitability index (PI).

- ROR is the most common measure of economic worth of a project or alternative and is calculated based on a PW or AW equation.

- The IRR method is one of the popular time-discounted measures of investment worth that is related to the NPV approach.

DEFINITION follows
Section 7.1: INTERPRETATION OF A RATE OF RETURN VALUE

DEFINITION

ROR is either the interest rate paid on the unpaid balance of a loan, or the interest rate earned on the unrecovered investment balance of an investment such that the final payment or receipt brings the terminal value to equal “0”.
Section 7.1: INTERPRETATION OF A RATE OF RETURN VALUE

◆ In rate of return problems you seek an unknown interest rate \((i^*)\) that satisfies the following:

\[
P_{W_{i^*}}(\text{+ cash flows}) - P_{W_{i^*}}(\text{- cash flows}) = 0
\]

◆ This means that the interest rate \(i^*\), is an unknown parameter and must be solved or approximated.
7.1 Unrecovered Investment Balance

◆ ROR is not the interest rate earned on the original loan amount or investment amount

◆ ROR is the interest rate earned/charged on the unrecovered investment balance of a loan or investment project which changes each time period

◆ The numerical value of ROR can range from -100% to infinity
7.1 Unrecovered Investment Balance

◆ Consider the following loan
◆ You borrow $1000 at 10% per year for 4 years
◆ You are to make 4 equal end of year payments to pay off this loan
◆ Your payments are:

\[ A = \$1000(A/P,10\%,4) = \$315.47 \]
# 7.1 The Loan Schedule

<table>
<thead>
<tr>
<th>Year</th>
<th>BOY Bal</th>
<th>Payment</th>
<th>Interest Amount</th>
<th>Recovered Amount</th>
<th>UnPaid Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1,000</td>
<td>0</td>
<td>--</td>
<td>---</td>
<td>$1,000</td>
</tr>
<tr>
<td>1</td>
<td>1,000</td>
<td>-315.47</td>
<td>100.00</td>
<td>-215.47</td>
<td>784.53</td>
</tr>
<tr>
<td>2</td>
<td>784.53</td>
<td>-315.47</td>
<td>78.45</td>
<td>-237.02</td>
<td>547.51</td>
</tr>
<tr>
<td>3</td>
<td>547.51</td>
<td>-315.47</td>
<td>54.75</td>
<td>-260.72</td>
<td>286.79</td>
</tr>
<tr>
<td>4</td>
<td>286.79</td>
<td>-315.47</td>
<td>28.68</td>
<td>-286.79</td>
<td>0</td>
</tr>
</tbody>
</table>
7.1 Unrecovered Investment Balance

◆ For this loan the unpaid loan balances at the end of each year are:

<table>
<thead>
<tr>
<th>Year (t)</th>
<th>Unpaid Loan Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1,000</td>
</tr>
<tr>
<td>1</td>
<td>784.53</td>
</tr>
<tr>
<td>2</td>
<td>547.51</td>
</tr>
<tr>
<td>3</td>
<td>286.79</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

◆ The ULB$_t = 4$ is exactly 0 at a 10% rate

Unpaid loan balance is now “0” at the end of the life of the loan
7.1 Reconsider the following

◆ Assume you invest $1000 over 4 years
◆ The investment generates $315.47/year
◆ Draw the cash flow diagram

```
0             1                2                3               4
P=$-1,000

A = +315.47
```
7.1 Investment Problem

◆ What interest rate equates the future positive cash flows to the initial investment?

◆ We can state:

\[ \$1000 = 315.47(P/A, i^*, 4) \]

\[ \text{Where } i^* \text{ is the unknown interest rate that makes the } PW(+) = PW(-) \]
7.1 Investment Problem

◆ $1000 = 315.47(P/A, i*,4)

◆ Solve the above for the $i^*$ rate

◆ $(P/A, i^*,4) = 1000/315.47 = 3.16987$

◆ Given $n = 4$ what value of $i^*$ yields a $P/A$ factor value = 3.16987?

◆ Interest Table search yields $i^* = 10\%$

◆ Just like the loan problem, we can calculate the unrecovered investment balances that are similar to the unpaid loan balances
7.1 Unrecovered Investment Balances (UIB)

We set up the following table:

<table>
<thead>
<tr>
<th>t</th>
<th>C.F(t)</th>
<th>Future Value for 1 year</th>
<th>$\text{UIB}_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1,000</td>
<td>----</td>
<td>-1,000</td>
</tr>
<tr>
<td>1</td>
<td>+315.47</td>
<td>-1000(1.10)+315.47=</td>
<td>-784.53</td>
</tr>
<tr>
<td>2</td>
<td>+315.47</td>
<td>784.53(1.10)+315.47=</td>
<td>-547.51</td>
</tr>
<tr>
<td>3</td>
<td>+315.47</td>
<td>547.51(1.10)+315.47=</td>
<td>-286.79</td>
</tr>
<tr>
<td>4</td>
<td>+315.47</td>
<td>286.79(1.10)+315.47=</td>
<td>0</td>
</tr>
</tbody>
</table>

- The unrecovered investment balances have been calculated at a 10% interest rate.
- Note, the investment is fully recovered at the end of year 4.
- The UIB = 0 at a 10% interest rate.
7.1 UIB’s for the Example

See Figure 7.1

<table>
<thead>
<tr>
<th>t</th>
<th>C.F(t)</th>
<th>UIB&lt;sub&gt;t&lt;/sub&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1,000</td>
<td>-1,000</td>
</tr>
<tr>
<td>1</td>
<td>+315.47</td>
<td>-784.53</td>
</tr>
<tr>
<td>2</td>
<td>+315.47</td>
<td>-547.51</td>
</tr>
<tr>
<td>3</td>
<td>+315.47</td>
<td>-286.79</td>
</tr>
<tr>
<td>4</td>
<td>+315.47</td>
<td>0</td>
</tr>
</tbody>
</table>

- The 10% rate is the only interest rate that will cause the UIB at the end of the project’s life to equal exactly “0”
- Note, all of UIB’s are negative at the 10% rate.
- This means that the investment is unrecovered throughout the life.
7.1 Pure Investment

◆ The basic definition of ROR is the interest rate that will cause the investment balance at the end of the project to exactly equal “0”

◆ If there is only one such interest rate that will cause this, the investment is said to be a “PURE” investment or, **Conventional investment**
7.1 UIB’s for the Example

◆ “Unrecovered” means that the investment balance for a given year is negative.

◆ If a project’s UIB’s are all negative (under-recovered) then that investment will possess one unique interest rate to cause the \( UIB_{t=n} \) to equal “0”
7.1 UIB’s for the Example

• This diagram depicts the unpaid loan balance

• At the ROR, the UIB will be exactly “0” at the end of the life of the project
7.1 ROR - Explained

◆ ROR is the interest rate earned on the unrecovered investment balances throughout the life of the investment.

◆ ROR is **not** the interest rate earned on the original investment.

◆ ROR ($i^*$) rate will also cause the NPV($i^*$) of the cash flow to = “0”.
Section 7.2: ROR using Present Worth or Annual Worth Equation

- **PW definition of ROR**
  
  \[ PW_{(-CF's)} = PW_{(+CF's)} \]

  \[ PW_{(-CF's)} - PW_{(+CF's)} = 0 \]

- **AW definition of ROR**
  
  \[ AW_{(-CF's)} = AW_{(+CF's)} \]

  \[ AW_{(-CF's)} - AW_{(+CF's)} = 0 \]
7.2 ROR using Present Worth

• Finding the ROR for most cash flows is a trial and error effort.

• The interest rate, $i^*$, is the unknown

• Solution is generally an approximation effort

• May require numerical analysis approaches
7.2 ROR using Present Worth

• See Figure 7.2

Assume you invest $1,000 at t = 0: Receive $500 @ t=3 and $1,500 at t = 5. What is the ROR of this project?
7.2 ROR using Present Worth

• Write a present worth expression, set equal to “0” and solve for the interest rate that satisfies the formulation.

\[ 1000 = 500(P/F, i*,3) + 1500(P/F, i*,5) \]

• Can you solve this directly for the value of \( i^* \)?
  • NO!

• Resort to trial and error approaches
7.2 ROR using Present Worth

\[ 1000 = 500(P/F, i^*,3) + 1500(P/F, i^*,5) \]

- Guess at a rate and try it
- Adjust accordingly
- Bracket
- Interpolate

\[ i^* \text{ approximately } 16.9\% \text{ per year on the unrecovered investment balances.} \]
7.2 Trial and Error Approach

• Iterative procedures require an initial guess for \( i^* \)

• One makes an educated first guess and calculates the resultant PV at the guess rate.
7.2 Trial and Error Approach

• If the NPV is not $= 0$ then another $i^*$ value is evaluated…. Until NPV “close” to “0”

• The objective is to obtain a negative PV for an $i^*$ guess value then.

• Adjust the $i^*$ value to obtain a positive PV given the adjusted $i^*$ value

• Then interpolate between the two $i^*$ values
7.2 Trial and Error Approach – Bracket “0”

• If the NPV is not $= 0$ then another $i^*$ value is evaluated

• A negative NPV generally indicates the $i^*$ value is too high

• A positive NPV suggests that the $i^*$ value was too low
7.2 ROR Criteria

• Determine the $i^*$ rate
• If $i^* \geq MARR$, accept the project
• If $i^* < MARR$, reject the project

Example 7.2
Excel supports ROR analysis

• **RATE(n,A,P,F) can be used when a time t = 0 investment (P) is made followed by “n” equal, end of period cash flows (A)**

• **This is a special case for annuities only**
7.2 Example 7.3 – In Excel

• Excel Setup for ROR

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$500,000</td>
</tr>
<tr>
<td>1</td>
<td>$10,000</td>
</tr>
<tr>
<td>2</td>
<td>$10,000</td>
</tr>
<tr>
<td>3</td>
<td>$10,000</td>
</tr>
<tr>
<td>4</td>
<td>$10,000</td>
</tr>
<tr>
<td>5</td>
<td>$10,000</td>
</tr>
<tr>
<td>6</td>
<td>$10,000</td>
</tr>
<tr>
<td>7</td>
<td>$10,000</td>
</tr>
<tr>
<td>8</td>
<td>$10,000</td>
</tr>
<tr>
<td>9</td>
<td>$10,000</td>
</tr>
<tr>
<td>10</td>
<td>$710,000</td>
</tr>
</tbody>
</table>

\[ \text{D19} = \text{Guess Value} \]

\[ \text{=IRR(D6:D16,D19)} \]
7.2 Example 7.2 Investment Balances

- Investment balances at the i* rate
- The time $t = 10$ balance = 0 at i*
- As it should!

<table>
<thead>
<tr>
<th>Year</th>
<th>Project Balances @ i* Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$500,000.00</td>
</tr>
<tr>
<td>1</td>
<td>-$515,784.79</td>
</tr>
<tr>
<td>2</td>
<td>-$532,383.60</td>
</tr>
<tr>
<td>3</td>
<td>-$549,838.40</td>
</tr>
<tr>
<td>4</td>
<td>-$568,193.34</td>
</tr>
<tr>
<td>5</td>
<td>-$587,494.83</td>
</tr>
<tr>
<td>6</td>
<td>-$607,791.70</td>
</tr>
<tr>
<td>7</td>
<td>-$629,135.26</td>
</tr>
<tr>
<td>8</td>
<td>-$651,579.50</td>
</tr>
<tr>
<td>9</td>
<td>-$675,181.19</td>
</tr>
<tr>
<td>10</td>
<td>$0.00</td>
</tr>
</tbody>
</table>
Section 7.3 Cautions when using the ROR Method

• Important Cautions to remember when using the ROR method......
7.3 Cautions when using the ROR Method No.

• Many real-world cash flows may possess multiple $i^*$ values

• More than one $i^*$ value that will satisfy the definitions of ROR

• If multiple $i^*$’s exist, which one, if any, is the correct $i^*$???
7.3 Cautions when using the ROR Method: Reinvestment Assumptions

• Reinvestment assumption for the ROR method is not the same as the reinvestment assumption for PW and AW

• PW and AW assume reinvestment at the MARR rate

• ROR assumes reinvestment at the i* rate

• Can get conflicting rankings with ROR vs. PV and AW
7.3 Cautions when using the ROR Method: Computational Difficulties

• ROR method is computationally more difficult than PW/AW

• Can become a numerical analysis problem and the result is an approximation

• Conceptually more difficult to understand
7.3 Cautions when using the ROR Method: Special Procedure for Multiple Alternatives

- For analysis of two or more alternatives using ROR one must resort to a different analysis approach as opposed to the PW/AW methods.

- For ROR analysis of multiple alternatives must apply an **incremental analysis** approach.
7.3 Cautions when using the ROR Method: ROR is more difficult!

• ROR is computationally more difficult
• But is a popular method with financial managers
• ROR is used internally by a substantial number of firms
• Suggest using PW/AW methods where possible
7.3 Valid Ranges for usable i* rates

Mathematically, i* rates must be:

\[-100\% < i^* \leq +\infty\]

- If an i* <= -100% this signals total and complete loss of capital.
- i*’s < -100% are not feasible and not considered
- One can have a negative i* value (feasible) but not less than –100%! 
Assignments and Announcements

- Assignments due at the beginning of next class:
  - Finish Reading chapter 7
Topics to Be Covered in Today’s Lecture

Section 7.4: Multiple Rates of Return
Section 7.5: Composite ROR Approach
Section 7.6: Rate of Return of a Bond Investment
Section 7.4: Multiple Rates of Return

- A class of ROR problems exists that will possess multiple i* values—called non-conventional or non-simple series
- Capability to predict the potential for multiple i* values
- Two Tests for Multiple i* values can be applied in sequences on the nonconventional series:
  1. Cash Flow Rule of Signs (aka as Descartes’ test)
  2. Cumulative Cash Flow Rule of Signs test (aka norstrom’s criterion)

Example follows:
7.4 Cash Flow Rule of Signs Test

- The total number of real values i*'s is always less than or equal to the number of sign changes in the original cash flow series.
- Follows from the analysis of a n-th degree polynomial
- A “0” value does not count as a sign change
- Example follows…
7.4 C.F. Rule of Signs example

• Consider Example 7.4

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2,000</td>
<td>+</td>
</tr>
<tr>
<td>1</td>
<td>-$500</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>-$8,100</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>$6,800</td>
<td>+</td>
</tr>
</tbody>
</table>

Result: 2 sign changes in the Cash Flow
7.4 Results: CF Rule of Signs Test

• Two sign changes in this example

• Means we can have a maximum of 2 real potential i* values for this problem

• Beware: This test is fairly weak and the second test must also be performed
7.4 Accumulated CF Sign Test (ACF)-Norstrom’s Test

• For the problem (Example 7.4) form the accumulated cash flow from the original cash flow.

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
<th>Accum. C.F</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2,000</td>
<td>$2,000</td>
</tr>
<tr>
<td>1</td>
<td>-$500</td>
<td>$1,500</td>
</tr>
<tr>
<td>2</td>
<td>-$8,100</td>
<td>-$6,600</td>
</tr>
<tr>
<td>3</td>
<td>$6,800</td>
<td>$200</td>
</tr>
</tbody>
</table>

Count sign changes here
## 7.4 Accumulated CF signs - Example

### A.C.F

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
<th>Accum. C.F</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2,000</td>
<td>$2,000</td>
</tr>
<tr>
<td>1</td>
<td>-$500</td>
<td>$1,500</td>
</tr>
<tr>
<td>2</td>
<td>-$8,100</td>
<td>-$6,600</td>
</tr>
<tr>
<td>3</td>
<td>$6,800</td>
<td>$200</td>
</tr>
</tbody>
</table>

2 sign changes in the ACF.
7.4 ACF Sign Test States:

• A **sufficient** but not necessary condition for a **single positive** i* value is:
  
  • The ACF value at year “N” is > 0
  
  • There is exactly one sign change in the ACF
7.4 ACF Test - continued

• If the value of the ACF for year “N” is “0” then an i* of 0% exists

• If the value of ACF for year “N” is > 0, this suggests an i* > 0

• If ACF for year N is < 0 there may exist one or more negative i* values – but not always
7.4 ACF Test - continued

• Alternatively:
• If the ACF in year “0” < 0
• And….

• One sign change in the ACF series then

• Have a unique i* value!
7.4 ACF Test - continued

• If the number of sign changes in the ACF is 2 or greater this strongly suggests that multiple rates of return exist.
### 7.4 Example 7.4 – ACF Sign Test

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
<th>Accum. C.F</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2,000</td>
<td>$2,000</td>
</tr>
<tr>
<td>1</td>
<td>-$500</td>
<td>$1,500</td>
</tr>
<tr>
<td>2</td>
<td>-$8,100</td>
<td>-$6,600</td>
</tr>
<tr>
<td>3</td>
<td>$6,800</td>
<td>$200</td>
</tr>
</tbody>
</table>

2 Sign Changes here

- **Strong evidence that we have multiple $i^* \text{ values}**
- **$ACF(t=3) = $200 > 0$ suggests positive $i^* \text{ (s)}$**
7.4 Excel Analysis of Ex. 7.4

We find two $i^*$ values:

\{7.47\%, 41.35\% \}

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2,000</td>
</tr>
<tr>
<td>1</td>
<td>-$500</td>
</tr>
<tr>
<td>2</td>
<td>-$8,100</td>
</tr>
<tr>
<td>3</td>
<td>$6,800</td>
</tr>
<tr>
<td>Sum</td>
<td>$200</td>
</tr>
</tbody>
</table>

- **First $i^*$ using a guess value of 0%**
  - ROR-Guess: 0%
  - ROR: 7.47%
  - NPV = $200.00

- **Second $i^*$ value using guess value of 30%**
  - ROR-Guess: 30%
  - ROR: 41.35%
### 7.4 Investment (Project) Balance

- Examine the Investment or Project balances at each $i^*$ rate

<table>
<thead>
<tr>
<th>Year</th>
<th>Project Balances @ $i^*$ Rate</th>
<th>Inv Bal $i$-rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2,000.00</td>
<td>7.47%</td>
</tr>
<tr>
<td>1</td>
<td>$1,649.36</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-$6,327.47</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$0.00</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Year</th>
<th>Project Balances @ $i^*$ Rate</th>
<th>Inv Bal $i$-rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2,000.00</td>
<td>41.35%</td>
</tr>
<tr>
<td>1</td>
<td>$2,327.04</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-$4,810.69</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$0.00</td>
<td></td>
</tr>
</tbody>
</table>
7.4 Investment Balances at both $i^*$’s

<table>
<thead>
<tr>
<th>Year</th>
<th>Project Balances @ $i^*$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2,000.00</td>
</tr>
<tr>
<td>1</td>
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</tr>
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<td>2</td>
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</tr>
<tr>
<td>3</td>
<td>$0.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Year</th>
<th>Project Balances @ $i^*$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2,000.00</td>
</tr>
<tr>
<td>1</td>
<td>$2,327.04</td>
</tr>
<tr>
<td>2</td>
<td>-$4,810.69</td>
</tr>
<tr>
<td>3</td>
<td>$0.00</td>
</tr>
</tbody>
</table>

- Terminal IB(7.47%) = 0
- Terminal IB(41.35%) = 0

**Both $i^*$’s yield terminal IB’s equal to 0!**
7.4 Investment Balances at both $i^*$’s

**Important Observations**

- The IB’s for the terminal year (3) both equal 0
- Means that the two $i^*$ values are valid ROR’s for this problem
- **Note:** The IB amounts are not all the same for the two $i^*$ values.
- **IB amounts are a function of the interest rate used to calculate the investment balances.**
7.4 PV Plot of 7.4

\[ i^* = 7.47\% \]

\[ i^* = 41.35\% \]
If the MARR is between the two $i^*$ values this investment would be rejected!
7.4 Comments on ROR

• Multiple $i^*$ values lead to interpretation problems

• If multiple $i^*$’s – which one, if any is the “correct” one to use in an analysis?

• Serves to illustrate the computational difficulties associated with ROR analysis

• Section 7.5 provides an alternative ROR approach – Composite ROR (C-ROR)
Section 7.5: Composite ROR Approach

Consider the following investment:

- Initial investment: $-10,000
- Year 2: $+$8,000
- Year 5: $+$9,000

Determine the ROR as...
### 7.5 Composite ROR Approach

#### Analysis reveals:

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$10,000</td>
</tr>
<tr>
<td>1</td>
<td>$0</td>
</tr>
<tr>
<td>2</td>
<td>$8,000</td>
</tr>
<tr>
<td>3</td>
<td>$0</td>
</tr>
<tr>
<td>4</td>
<td>$0</td>
</tr>
<tr>
<td>5</td>
<td>$9,000</td>
</tr>
<tr>
<td>Sum</td>
<td>$7,000</td>
</tr>
</tbody>
</table>

ROR-Guess = 0%

ROR = 16.82%

\[ i^* = 16.82\% / \text{year on the unrecovered investment balances over 5 years} \]
7.5 Composite ROR Approach: IB’s

**The Investment Balances at i* are:**

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
<th>Project Balances @ i* Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-$10,000</td>
<td>-$10,000.00</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-$11,681.59</td>
</tr>
<tr>
<td>2</td>
<td>$8,000</td>
<td>-$5,645.95</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-$6,595.36</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>-$7,704.43</td>
</tr>
<tr>
<td>5</td>
<td>$9,000</td>
<td>$0.00</td>
</tr>
<tr>
<td>Sum</td>
<td>$7,000</td>
<td></td>
</tr>
</tbody>
</table>

**ROR-Guess** 0%

**ROR** 16.82%

- All IB’s are negative for t = 0 – 4: IB(5) = 0
- Conventional (pure) investment
7.5 Composite ROR Approach

- $10,000

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>+$9,000</td>
</tr>
<tr>
<td>+$8,000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Question:** Is it reasonable to assume that the +8,000 can be invested forward at 16.82%?
7.5 Composite ROR Approach

• Remember ……

• ROR assumes reinvestment at the calculated i* rate

• What if it is not practical for the +$8,000 to be reinvested forward one year at 16.82%?
7.5 Reinvestment Rates

• Most firms can reinvest surplus funds at some conservative market rate of interest in effect at the time the surplus funds become available.

• Often, the current market rate is less than a calculated ROR value

• What then is the firm to do with the +$8,000 when it comes in to the firm?
7.5 Reinvesting

• Surplus funds must be put to good use by the firm

• These funds belong to the owners – not to the firm!

• Owners expect such funds to be put to work for benefit of future wealth of the owners
7.5 Composite ROR Approach

- Consider the following representation.
7.5 Composite ROR Approach

- Or, put in another context....

The Firm

Project borrows from the firm

Project

Project Lends back to the firm
7.5 Composite ROR Approach

• Or, put in another context....

Project borrows from the firm
At the i* rate (16.82%)

Project Lends back to the firm
But can the firm reinvest these funds at 16.82%? Probably not!
7.5 Composite ROR Approach

• So, we may have to consider a reinvestment rate that is closer to the current market rate for reinvestment of the $8,000 for the next time period(s)

• Assume a reasonable market rate is say, 8% per year.

• Call this rate an external rate - c
7.5 The external rate - $c$

- The external interest rate – $c$, is a rate that the firm can reinvest surplus funds for at least one time period at a time.
- $c$ is often set to equal the firm’s current MARR rate
7.5 Composite ROR Approach

• Thus, a procedure has been developed that will determine the following:
  • Find i* given c – if multiple ROR’s exist.
  • For multiple i*’s in a problem, the analysis determines a single i* given c
  • Denoted i*/c or, i’
  • i’ is called the composite rate = i*/c

The “/” is read “given” i.e., i* given a value for “c”
7.5 Composite ROR Approach

- Finding $i'$ is a much more involved process
- Prior to digital computers, only very small ($N \leq \text{say } 4-5 \text{ time periods}$) could be manually evaluated
- Requires a recursive analysis best left to a computer program and spreadsheet
- Example 7.6 illustrates a manual approach
## Example 7.6

**Cash Flow is:**

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2,000</td>
</tr>
<tr>
<td>1</td>
<td>-$500</td>
</tr>
<tr>
<td>2</td>
<td>-$8,100</td>
</tr>
<tr>
<td>3</td>
<td>$6,800</td>
</tr>
</tbody>
</table>
7.5 Ex. 7-5 Multiple i*’s

- $i^*_1 = 7.468; \; i^*_2 = 41.35\%$

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
<th>ROR-Guess</th>
<th>ROR</th>
<th>NPV</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2,000</td>
<td>0%</td>
<td>$200.00</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-$500</td>
<td>7.468%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-$8,100</td>
<td>30%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$6,800</td>
<td>41.35%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
7.5 Investment Balances are:

- **IB(7.468%)**

<table>
<thead>
<tr>
<th>Inv Bal i-rate</th>
<th>7.468%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year</td>
<td>Project Balances @ i* Rate</td>
</tr>
<tr>
<td>0</td>
<td>$2,000.00</td>
</tr>
<tr>
<td>1</td>
<td>$1,649.36</td>
</tr>
<tr>
<td>2</td>
<td>-$6,327.47</td>
</tr>
<tr>
<td>3</td>
<td>$0.00</td>
</tr>
</tbody>
</table>

- **IB(41.352%)**

<table>
<thead>
<tr>
<th>Inv Bal i-rate</th>
<th>41.352%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year</td>
<td>Project Balances @ i* Rate</td>
</tr>
<tr>
<td>0</td>
<td>$2,000.00</td>
</tr>
<tr>
<td>1</td>
<td>$2,327.04</td>
</tr>
<tr>
<td>2</td>
<td>-$4,810.69</td>
</tr>
<tr>
<td>3</td>
<td>$0.00</td>
</tr>
</tbody>
</table>
7.5 Project Balances @ 41.35%

Project Balances

Over-recovered

Under-recovered

Years

PB's - $

EGR2302-Engineering Economics
Al Akhawayn University
7.5 Project Balances @ 7.468%%

-8,000
-6,000
-4,000
-2,000
 0
 2,000
 4,000
 0 1 2 3

PB’s - $

$0

Over-recovered

Under-recovered

Years

Project Balances

7.5 Project Balances @ 7.468%%
7.5 Assume you **Reinvest** at 20%

- \( c = 20\% \)
- Positive IB’s are reinvested at 20% -
- **Not at the computed** \( i^* \) **rate**
- **Now, what is the modified ROR –** \( i' \)?
- **Must perform a recursive analysis**
7.5 Recursive IB’s are….

• Let $I_b_j = F_j$

• $F_0 = +2000$ (> 0; invest at 20%)

• $F_1 = 2000(1.20) - 500 = +1900$
  • $+1900 > 0$; invest at 20%

• $F_2 = 1900(1.20) - 8100 = -5820$
  • $-5820 < 0$; invest at $i'$ rate

Under-recovered
7.5 Recursive IB’s are….

• $F_2 = 1900(1.20) - 8100 = -5820$

-5820 < 0; invest at $i'$ rate

• $F_3 = -5820(1+i') + 6800$

• Since $N = 3$, $F_3 = 0$

• Solve; $-5820(1+i') + 6800 = 0$

Repeated from the previous slide for clarity
7.5 Single conditional ROR value

• \(-5820(1+i') + 6800 = 0\)
• \(i' = [6800/5820] -1\)
• \(i' = 16.84\%\) given \(c = 20\%\)
• If MARR = 20\% and \(i' = 16.84\%\) this investment would be rejected
• Reduced the problem to a single conditional ROR value.
7.5 What if $c = i^*_1$ (7.47%)?

• Here, we examine what happens IF we assume the reinvestment rate, $c$, equals one of the computed $i^*$ values.

• This approach assumes that the reinvestment rate, $c$, is one of the $i^*$ rates.

• Recursive calculations follow….
7.5 Recursive IB’s are….(@7.47%)

- Let \( IB_j = F_j \)
- \( F_0 = +2000 \) (> 0; invest at 7.47%)
- \( F_1 = 2000(1.0747) - 500 = +1649.40 \)
  - >0: Over-recovered balance
- \( F_2 = +1649.40(1.074) - 8100 = -6327.39 \)
  - Now, under-recovered balance
7.5 Recursive IB’s are....

• $F_2 = +1649.40(1.074) - 8100 = -6327.39$

• $F_3 = -6327(1+i') + 6800$

• Since $N = 3$, $F_3$ must = 0 (terminal IB condition)

• Solve; $-6327(1+i') + 6800 = 0$
7.5 Single conditional ROR value

• Must solve:
  • \(-6327.39(1+i') + 6800 = 0\)
  • \((1+i') = \frac{6800}{6327.39}\)
  • \(i' = 0.0747 = 7.47\%\)

• Given \(c = \text{one of the } i^*\)'s, the \(i'\) will equal the \(i^*\) used as the reinvestment rate!
Section 7.6: Rate of Return of a Bond Investment

• Examples 7.8 and 7.9
Assignments and Announcements

- Assignments due at the beginning of next class:
  - Do the online quizzes for Chapter 7
  - Homework for chapter 7: 7.10, 7.25, 7.26, 7.31.