Chapter 4: Nominal and Effective Interest Rates

Session 9-10-11
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Topics to Be Covered in Today’s Lecture

- Section 4.1: Nominal and Effective Interest Rates statements
- Section 4.2: Effective Annual Interest Rates
Section 4.1
NOMINAL & EFFECTIVE RATES

- Review Simple Interest and Compound Interest (from Chapter 1)
- Compound Interest –
  - Interest computed on Interest
  - For a given interest period
- The time standard for interest computations – One Year
- One Year: Can be segmented into:
  - 365 days
  - 52 Weeks
  - 12 Months
  - One quarter: 3 months – 4 quarters/year
- Interest can be computed more frequently than one time a year
4.1 Common Compounding Frequencies

- Interest May be computed (compounded):
  - Annually – One time a year (at the end)
  - Every 6 months – 2 times a year (semi-annual)
  - Every quarter – 4 times a year (quarterly)
  - Every Month – 12 times a year (monthly)
  - Every Day – 365 times a year (daily)
  - …
  - Continuous – infinite number of compounding periods in a year.
4.1 Quotation of Interest Rates

- Interest rates can be quoted in more than one way.
- Example:
  - Interest equals “5% per 6-months”
  - Interest is “12%” (12% per what?)
  - Interest is 1% per month
  - “Interest is “12.5% per year, compounded monthly”
- Thus, one must “decipher” the various ways to state interest and to calculate.
4.1 Two Common Forms of Quotation

- Two types of interest quotation
  - 1. Quotation using a Nominal Interest Rate
  - 2. Quoting an Effective Periodic Interest Rate

- Nominal and Effective Interest rates are common in business, finance, and engineering economy

- Each type must be understood in order to solve various problems where interest is stated in various ways.
4.1 Notion of a Nominal Interest Rate

• A *Nominal* Interest Rate, $r$.

• Definition:

A Nominal Interest Rate, $r$, is an interest Rate that does not include any consideration of compounding.

Nominal means, “in name only”, not the real rate in this case.
4.1 Quoting a Nominal Interest Rate

- Interest rates may be quoted (stated – communicated) in terms of a nominal rate.
- You will see there are two ways to quote an interest rate:
  - 1. Quote the Nominal rate
  - 2. Quote the true, effective rate.
- For now – we study the nominal quotation.
4.1 Definition of a Nominal Interest Rate

• Mathematically we have the following definition:

\[ r = (\text{interest rate per period})(\text{No. of Periods}) \]

Examples Follow.....
4.1 Examples – Nominal Interest Rates

• 1.5% per month for 24 months
  – Same as: \((1.5\%) (24) = 36\%\) per 24 months

• 1.5% per month for 12 months
  – Same as \((1.5\%) (12\text{ months}) = 18\%/\text{year}\)

• 1.5% per month for 6 months
  – Same as: \((1.5\%) (6\text{ months}) = 9\%/6\text{ months or semiannual period}\)

• 1% per week for 1 year
  – Same as: \((1\%) (52\text{ weeks}) = 52\%\) per year
4.1 Nominal Rates…..

• A nominal rate (so quoted) do not reference the frequency of compounding. They all have the format “$r\% \text{ per time period}$”

• Nominal rates can be misleading

• We need an alternative way to quote interest rates…. 

• The true *Effective Interest Rate* is then applied…. 
4.1 The Effective Interest Rate (EIR)

• When so quoted, an Effective interest rate is a true, periodic interest rate.
• It is a rate that applies for a stated period of time
• It is conventional to use the year as the time standard
• So, the EIR is often referred to as the Effective Annual Interest Rate (EAIR)
4.1 The EAIR

• Example:
  – “12 per cent compounded monthly”

• Pick this statement apart:
  – 12% is the nominal rate
  – “compounded monthly” conveys the frequency of the compounding throughout the year
  – This example: 12 compounding periods within a year.
4.1 The EAIR and the Nominal Rate

- The EAIR adds to a nominal rate by informing the user of the frequency of compounding within a year.
- Notation:
- It is conventional to use the following notation for the EAIR
  - “\(i_a\)” or,
  - “\(i\)”
- The EAIR is an extension of the nominal rate – “\(r\)”
4.1 Focus on the Differences

• Nominal Rates:
  – Format: “r% per time period, t”
  – Ex: 5% per 6-months”

• Effective Interest Rates:
  – Format: “r% per time period, compounded ‘m’ times a year.
  – ‘m’ denotes or infers the number of times per year that interest is compounded.
  – Ex: 18% per year, compounded monthly
4.1 Which One to Use: “r” or “i”?

• Some problems may state only the nominal interest rate.

• **Remember:** *Always apply the Effective Interest Rate in solving problems.*

• Published interest tables, closed-form time value of money formula, and spreadsheet function assume that only Effective interest is applied in the computations.
4.1 Time Based Units Associated with Interest Rate Statements

- Time Period, \( t \) - *Interest rates are stated as \( \% \) per time period.* (\( t \) usually in years)
- Compounding Period, (CP) - *Length of time between compounding operations.*
- Compounding Frequency - Let “m” represent the number of times that interest is computed (compounded) within time period “\( t \)”. 

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4.1 Effective Rate per CP

- The Effective interest Rate per compounding period, CP is:

\[
i_{\text{effective per CP}} = \frac{r\%/\text{time period } t}{m \text{ compounding periods/t}}
\]
4.1 Example:

- Given:

  \[ r = 9\% \text{ per year, compounded monthly} \]

  \[
  \text{Effective Monthly Rate:} \\
  \frac{0.09}{12} = 0.0075 = 0.75\%/\text{month}
  \]

Here, “m” counts months so, \( m = 12 \) compounding periods within a year.
4.1 Example 4.1
• Given, “9% per year, compounded quarterly”

One Year: Equals 4 Quarters

CP equals a quarter (3 – months)
4.1 Example 4.1 (9%/yr: Compounded quarterly)

- Given, “9% per year, compounded quarterly”

<table>
<thead>
<tr>
<th>Qtr. 1</th>
<th>Qtr. 2</th>
<th>Qtr. 3</th>
<th>Qtr. 4</th>
</tr>
</thead>
</table>

What is the Effective Rate per Quarter?

- \( i_{\text{Qtr.}} = 0.09/4 = 0.0225 = 2.25\%/\text{quarter} \)
- 9% rate is a nominal rate;
- The 2.25% rate is a true effective monthly rate
4.1 Example 4.1 (9%/yr: Compounded quarterly)

- Given, “9% per year, compounded quarterly”

| Qtr. 1: 2.25% | Qtr. 2: 2.25% | Qtr. 3: 2.25% | Qtr. 4: 2.25% |

The effective rate (true rate) per quarter is 2.25% per quarter
4.1 Statement: 9% compounded monthly

- \( r = 9\% \) (the nominal rate).
- “compounded monthly means “\( m \)” =12.
- The true (effective) monthly rate is:
  - \( \frac{0.09}{12} = 0.0075 = 0.75\% \text{ per month} \)

One Year Duration (12 months)
4.1 Statement: 4.5% per 6 months – compounded weekly

- Nominal Rate: 4.5%.
- Time Period: 6 months.
- Compounded weekly:
  - Assume 52 weeks per year
  - 6-months then equal $52/2 = 26$ weeks per 6 months
- The true, effective weekly rate is:
  - $(0.045/26) = 0.00173 = 0.173\% \text{ per week}$
4.1 Table 4.1

- It can be unclear as to whether a stated rate is a nominal rate or an effective rate.
- Three different statements of interest follow........
4.1 Varying Statements of Interest Rates

- “8% per year, compounded quarterly”
  - Nominal rate is stated: 8%
  - Compounding Frequency is given
    - Compounded quarterly
    - True quarterly rate is \( \frac{0.8}{4} = 0.02 = 2\% \text{ per quarter} \)

Here, one must calculate the effective quarterly rate!
4.1 Effective Rate Stated

• “Effective rate = 8.243% per year, compounded quarterly:
  – No nominal rate given (must be calculated)
  – Compounding periods – m = 4

• No need to calculate the true effective rate!
  – It is already given: 8.243% per year!
4.1 Only the interest rate is stated

- “8% per year”.

- Note:
  - No information on the frequency of compounding.
  - Must assume it is for one year!
  - “m” is assumed to equal “1”.

- Assume that “8% per year” is a true, effective annual rate!
  - No other choice!
Section 4.2
Effective Annual Interest Rates

• Here, we show how to calculate true, effective, annual interest rates.

• We assume the year is the standard of measure for time.

• The year can be comprised of various numbers of compounding periods (within the year).
4.2 Typical Compounding Frequencies

• Given that one year is the standard:
  – \( m = 1 \): compounded annually (end of the year);
  – \( m = 2 \): semi-annual compounding (every 6 months);
  – \( m = 4 \): quarterly compounding;
  – \( m = 12 \): monthly compounding;
  – \( m = 52 \): weekly compounding;
  – \( m = 365 \): daily compounding;

• Could keep sub-dividing the year into smaller and smaller time periods.
4.2 Calculation of the EAIR

- EAIR – “the Effective Annual Interest Rate”.
- The EAIR is the true, annual rate given a frequency of compounding within the year.
- We need the following notation…….
4.2 EAIR Notation

- $r$ = the nominal interest rate per year.
- $m$ = the number of compounding periods within the year.
- $i$ = the effective interest rate per compounding period ($r/m$)
- $i_a$ or $i_e$ = the true, effective annual rate given the value of $m$. 
4.2 Derivation of the EAIR relationship

• Assume $1 of principal at time $t = 0$.
• Conduct a period-by-period Future Worth calculation.
• Notation “problem”.
  – At times, “$i$” is used in place of “$i_e$” or “$i_a$”.
  – So, “$i$” can also represent the true effective annual interest rate!
4.2 Deriving the EAIR…

- Consider a one-year time period.

Invest $1 of principal at time $t = 0$ at interest rate $i$ per year.

One year later, $F = P(1+ia)^1$
4.2 Deriving the EAIR...

- Interest could be compounded more than one time within the year!

\[ F = P(1 + ia)^m \]

Assume the one year is now divided into "m" compounding periods.

Replace "i" with "i_a" since m now > 1.
4.2 Rewriting….

- $F = P + P(i_a)$
- Now, the rate $i$ per CP must be compounded through all “m” periods to obtain $F_1$
- Rewrite as:
  - $F = P + P(i_a) = P(1 + i_a)$
  - $F = P(1 + i)^m$
4.2 Two similar expressions for F

- We have two expressions for F;
- \( F = P(1 + i_a) \);
- \( F = P(1 + i)^m \);
- Equate the two expressions;
- \( P(1 + i_a) = P(1 + i)^m \);
- \( R(1 + i_a) = R(1 + i)^m \);

Solve \( i_a \) in terms of “i”.
4.2 Expression for $i_a$

- Solving for $i_a$ yields:

\[
1 + i_a = (1+i)^m \quad (1)
\]

\[
i_a = (1 + i)^m - 1 \quad (2)
\]

If we start with a nominal rate, "r" then....
4.2 The EAIR is……

- Given a nominal rate, “r”
- \( i_{\text{Compounding period}} = \frac{r}{m} \)
- The EAIR is calculated as;

\[
\text{EAIR} = (1 + \frac{r}{m})^m - 1. \tag{3}
\]

Or, \( i_{\text{Compounding period}} = (1 + i_a)^{1/m} - 1 \)

Then: Nominal rate – “r” = \((i)(m)\) \(\tag{4}\)
4.2 Example: EAIR given a nominal rate.

- Given: interest is 8% per year compounded quarterly”.
- What is the true annual interest rate?
- Calculate:

\[
EAIR = (1 + 0.08/4)^4 - 1
\]

\[
EAIR = (1.02)^4 - 1 = 0.0824 = 8.24\%/year
\]
4.2 Example: “18%/year, comp. monthly”

- What is the true, effective annual interest rate?
  \[ r = \frac{0.18}{12} = 0.015 = 1.5\% \text{ per month}. \]

1.5% per month is an effective monthly rate.

The effective annual rate is:

\[ (1 + \frac{0.18}{12})^{12} - 1 = 0.1956 = 19.56\%/\text{year} \]
4.2 Previous Example

• “18%, c.m. (compounded monthly)

• Note:
  – Nominal Rate is 18%;
  – The true effective monthly rate is 1.5%/month;
  – The true effective annual rate is 19.56%/year.

• One nominal rate creates 2 effective rates!
  – Periodic rate and an effective annual rate.
4.2 EAIR’s for 18%

- $m = 1$
  - $EAIR = (1 + 0.18/1)^1 - 1 = 0.18 \text{ (18\%)}$

- $m = 2$ (semiannual compounding)
  - $EAIR = (1 + 0.18/2)^2 - 1 = 18.810\%$

- $m = 4$ (quarterly compounding)
  - $EAIR = (1 + 0.18/4)^4 - 1 = 19.252\%$

- $m = 12$ (monthly compounding)
  - $EAIR = (1 + 0.18/12)^{12} - 1 = 19.562\%$

- $m = 52$ (weekly compounding)
  - $EAIR = (1 + 0.18/52)^{52} - 1 = 19.684\%$
4.2 Continuing for 18%.....

- \( m = 365 \) (daily compounding).
  - \( \text{EAIR} = (1 + 0.18/365)^{365} - 1 = 19.714\% \)

- \( m = 365(24) \) (hourly compounding).
  - \( \text{EAIR} = (1 + 0.18/8760)^{8760} - 1 = 19.72\% \)

- Could keep subdividing the year into smaller time periods.

- Note: There is an apparent limit as “\( m \)” gets larger and larger…called \textit{continuous} compounding.
### 4.2 Example: 12% Nominal

<table>
<thead>
<tr>
<th>No. of Comp. Per.</th>
<th>EAIR (Decimal)</th>
<th>EAIR (per cent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual</td>
<td>0.1200000</td>
<td>12.00000%</td>
</tr>
<tr>
<td>semi-annual</td>
<td>0.1236000</td>
<td>12.36000%</td>
</tr>
<tr>
<td>Quarterly</td>
<td>0.1255088</td>
<td>12.55088%</td>
</tr>
<tr>
<td>Bi-monthly</td>
<td>0.1261624</td>
<td>12.61624%</td>
</tr>
<tr>
<td>Monthly</td>
<td>0.1268250</td>
<td>12.68250%</td>
</tr>
<tr>
<td>Weekly</td>
<td>0.1273410</td>
<td>12.73410%</td>
</tr>
<tr>
<td>Daily</td>
<td>0.1274746</td>
<td>12.74746%</td>
</tr>
<tr>
<td>Hourly</td>
<td>0.1274959</td>
<td>12.74959%</td>
</tr>
<tr>
<td>Minutes</td>
<td>0.1274968</td>
<td>12.74968%</td>
</tr>
<tr>
<td>seconds</td>
<td>0.1274969</td>
<td>12.74969%</td>
</tr>
</tbody>
</table>

12% nominal for various compounding periods
Assignments and Announcements

- Assignments due at the beginning of next class:
  - Read Sections 4.2, 4.3, 4.4, 4.5,
Topics to Be Covered in Today’s Lecture

- Section 4.3: Payment Period (PP)
- Section 4.4: Equivalence: Comparing PP to CP
- Section 4.5: Single Amounts: CP \( \geq \) PP
- Section 4.6: Series Analysis – PP \( \geq \) CP
- Section 4.7: Single Amounts/Series with PP \( < \) CP
Section 4.3: Payment Period (PP)

• Recall:
  – CP is the “compounding period”

• PP is now introduced:
  – PP is the “payment period”

• Why “CP” and “PP”?
  – Often the frequency of depositing funds or making payments does not coincide with the frequency of compounding.
4.3 Comparisons:

• Example 4.4

Three different interest charging plans. Payments are made on a loan every 6 months. Three interest plans are presented:

1. 9% per year, c.q. (compounded quarterly).
2. 3% per quarter, (compounded quarterly).
3. 8.8% per year, c.m. (compounded monthly)

Which Plan has the lowest annual interest rate?
4.3 Comparing 3 Plans: Plan 1

- 9% per year, c.q.
- Payments made every 6 months.

9%, c.q. = 0.09/4 = 0.045 per 3 months = 2.25% per 3 months

Rule: The interest rate must match the payment period!
4.3 The Matching Rule

- Again, the interest must be consistent with the payment period!
- We need a 6-month effective rate and then calculate the 1 year true, effective rate!
- To compare the 3 plans:
  - Compute the true, effective 6-month rate or,
  - Compute the true effective 1 year rate.
  - Then one can compare the 3 plans!
4.3 Comparing 3 Plans: Plan 1

- 9% per year, c.q. = 2.25%/quarter
- Payments made every 6 months.

True 6-month rate is:

\[(1.0225)^2 - 1 = 0.0455 = 4.55\% \text{ per 6-months}\]

EAIR = \[(1.0225)^4 - 1 = 9.31\% \text{ per year}\]
4.3 Plan 2

• 3% per quarter, c.q.
• Effective=3%/quarter
• Find the EIR for 6-months
• Calculate:
  – For a 6-month effective interest rate -
  – \((1.03)^2 - 1 = 0.0609 = \textbf{6.09\% per 6-months}\)
  – Or, for a 1 year effective interest rate -
  – \((1.03)^4 - 1 = \textbf{12.55\%/year}\)
4.3 Plan 3:” 8.8% per year, c.m.”

- “r” = 8.8%
- “m” = 12
- Payments are twice a year
- 6-month nominal rate = 0.088/2 = 4.4%/6-months (“r” = 0.044)
- \[ EIR_{\text{6-months}} = (1 + 0.044/6)^6 - 1 = (1.0073)^6 - 1 = 4.48\%/\text{6-months} \]
- \[ EIR_{\text{12-months}} = (1 + 0.088/12)^{12} - 1 = 9.16\%/\text{year} \]
4.3 Summarizing the 3 plans….

<table>
<thead>
<tr>
<th>Plan No.</th>
<th>6-month</th>
<th>1-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.55%</td>
<td>9.31%</td>
</tr>
<tr>
<td>2</td>
<td>6.09%</td>
<td>12.55%</td>
</tr>
<tr>
<td>3</td>
<td>4.48%</td>
<td>9.16%</td>
</tr>
</tbody>
</table>

Plan 3 presents the lowest interest rate.
4.3 Can be confusing ???

- The 3 plans state interest differently.
- Difficult to determine the best plan by mere inspection.
- Each plan must be evaluated by:
  - Calculating the true, effective 6-month rate
  Or,
  - Calculating the true, effective 12 month, (1 year) true, effective annual rate.
  - Then all 3 plans can be compared using the EIR or the EAIR.
Section 4.4:
Equivalence: Comparing PP to CP

• Reality:
  – PP and CP’s do not always match up;
  – May have monthly cash flows but…
  – Compounding period different than monthly.

• Savings Accounts – for example;
  – Monthly deposits with,
  – Quarterly interest earned or paid;
  – They don’t match!

• Make them match! (by adjusting the interest period to match the payment period.)
## Situations

<table>
<thead>
<tr>
<th>Situation</th>
<th>Text Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>PP = CP</td>
<td>Sections 4.5 and 4.6</td>
</tr>
<tr>
<td>PP &gt; CP</td>
<td>Sections 4.5 and 4.6</td>
</tr>
<tr>
<td>PP &lt; CP</td>
<td>Section 4.7</td>
</tr>
</tbody>
</table>
Section 4.5
Single Amounts: PP $\geq$ CP

Example 1:

- “r” = 15%, c.m. (compounded monthly)
- Let $P = \$1500.00$
- Find $F$ at $t = 2$ years.
- $15\%$ c.m. = $0.15/12 = 0.0125 = 1.25\%/\text{month}$.
- $n = 2$ years OR 24 months
- Work in months or in years
4.5 Single Amounts: PP >= CP

- Approach 1. (n relates to months)
- State:
  - $F_{24} = \$1,500(F/P,0.15/12,24)$;
  - $i/\text{month} = 0.15/12 = 0.0125$ (1.25%);
  - $F_{24} = \$1,500(F/P,1.25\%,24)$;
  - $F_{24} = \$1,500(1.0125)^{24} = \$1,500(1.3474)$;
  - $F_{24} = \$2,021.03$. 
4.5 Single Amounts: CP >= CP

• Approach 2. (n relates to years)
• State:
  – \( F_2 = 1,500(F/P,i\%,2) \);
  – Assume \( n = 2 \) (years) we need to apply an annual effective interest rate.
  – \( i/\text{month} = 0.0125 \)
  – \( \text{EAIR} = (1.0125)^{12} - 1 = 0.1608 \) (16.08%)
  – \( F_2 = 1,500(F/P,16.08\%,2) \)
  – \( F_2 = 1,500(1.1608)^2 = $2,021.19 \)
  – **Slight roundoff compared to approach 1**
4.5 Example 2.

Consider

\[ r = 12\% \text{/yr, c.s.a.} \]

\[
\begin{align*}
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8 & \quad 9 & \quad 10 \\
\$1,000 & \quad & \$3,000 & \quad & \$1,500 & \quad \\
\end{align*}
\]

\[ F_{10} = ? \]

Suggest you work this in 6-month time frames.
Count “n” in terms of “6-month” intervals.
4.5 Example 2.

- Renumber the time line

\[ r = 12\% / \text{yr, c.s.a.} \]

\[
\begin{align*}
0 & \quad 2 & \quad 4 & \quad 6 & \quad 8 & \quad 10 & \quad 12 & \quad 14 & \quad 16 & \quad 18 & \quad 20 \\
\$1,000 & & & & & & & & & & \\
\$1,500 & & & & & & & & & & \\
\$3,000 & & & & & & & & & & \\
\end{align*}
\]

\[ F_{10} = ? \]

\[ i/6 \text{ months} = 0.12/2 = 6\%/6 \text{ months}; n \text{ counts 6-month time periods} \]
4.5 Example 2.

- **Compound Forward**

\[ r = 12\%/yr, \text{ c.s.a.} \]

\[
F_{20} = \$1,000(F/P,6\%,20) + \$3,000(F/P,6\%,12) + \\
\$1,500(F/P,6\%,8) = \$11,634
\]
4.5 Example 2.
Let n count years....

- Compound Forward

$r = 12\%/yr, \text{ c.s.a.}$

IF n counts years, interest must be an annual rate.

$\text{EAIR} = (1.06)^2 - 1 = 12.36\%$

Compute the FV where n is years and $i = 12.36\%$!
Section 4.6
Series Analysis – PP \( \geq \) CP

- Find the effective “\( i \)” per payment period.
- Determine “\( n \)” as the total number of payment periods.
- “\( n \)” will equal the number of cash flow periods in the particular series.
- Example follows…..
4.6 Series Example

Consider:

$A = 500$ every 6 months

Find $F_7$ if “$r$” = 20%/yr, c.q. $(PP > CP)$

We need $i$ per 6-months – effective.

$i_{\text{6-months}} = \text{adjusting the nominal rate to fit.}$
4.6 Series Example

• Adjusting the interest
• \( r = 20\% \), c.q.
• \( i/\text{qtr.} = \frac{0.20}{4} = 0.05 = 5\%/\text{qtr.} \)
• 2-qtrs in a 6-month period.
• \( i_{\text{6-months}} = (1.05)^2 - 1 = 10.25\%/\text{6-months} \).
• Now, the interest matches the payments.
• \( F_{\text{year 7}} = F_{\text{period 14}} = 500(\text{F/A,10.25\%,14}) \)
• \( F = 500(28.4891) = 14,244.50 \)
4.6 This Example: Observations

- Interest rate must match the frequency of the payments.
- In this example – we need effective interest per 6-months: Payments are every 6-months.
- The effective 6-month rate computed to equal 10.25% - un-tabulated rate.
- Calculate the F/A factor or interpolate.
- Or, use a spreadsheet that can quickly determine the correct factor!
4.6 This Example: Observations

• Do not attempt to adjust the payments to fit the interest rate!

• This is Wrong!

• At best a gross approximation – do not do it!

• This type of problem almost always results in an un-tabulated interest rate
  – You have to use your calculator to compute the factor or a spreadsheet model to achieve exact result.
Section 4.7
Single Amounts/Series with PP < CP

- This situation is different than the last.
- Here, PP is less than the compounding period (CP).
- Raises questions?
- Issue of interperiod compounding
- An example follows.
4.7 Interperiod Compounding Issues

- Consider a one-year cash flow situation.
- Payments are made at end of a given month.
- Interest rate is “r = 12%/yr, c.q.”
4.7 Interperiod Compounding

- $r = 12\% /\text{yr. c.q.}$

Note where some of the cash flow amounts fall with respect to the compounding periods!
4.7 Take the first $200 cash flow

- Will any interest be earned/owed on the $200 since interest is compounded at the end of each quarter?

The $200 is at the end of month 2 and will it earn interest for one month to go to the end of the first compounding period?

The last month of the first compounding period. Is this an interest-earning period?
4.7 Interperiod Issues

• The $200 occurs 1 month before the end of compounding period 1.

• Will interest be earned or charged on that $200 for the one month?

• If not then the revised cash flow diagram for all of the cash flows should look like.....
4.7 No interperiod compounding

- Revised CF Diagram

All negative CF’s move to the end of their respective quarters and all positive CF’s move to the beginning of their respective quarters.
4.7 No interperiod compounding

- Revised CF Diagram

Now, determine the future worth of this revised series using the F/P factor on each cash flow.
4.7 Final Results: No interperiod Comp.

- With the revised CF compute the future worth.

“r” = 12%/year, compounded quarterly

“i” = 0.12/4 = 0.03 = 3% per quarter

\[
F_{12} = [-150(F/P,3\%,4) – 200(F/P,3\%,3) + (-175+90)(F/P,3\%,2) + 165(F/P,3\%,1) – 50] \\
= -$357.59
\]
4.7 Interperiod Compounding

- \( r = 12\% \text{/yr. c.q.} \)

The cash flows are not moved and equivalent P, F, or A values are determined using the effective interest rate per payment period.
4.7 Interperiod Compounding

- \( r = 12\% \text{/yr. c.q.} \)

If the inter-period compounding is earned, then we should compute the effective interest rate per compounding period.

\( i = 3\% \) is the effective quarterly rate

\[
I_{\text{monthly}} = (1+i)^{1/3}-1 = (1.03)^{1/3}-1 = 0.99\%
\]
Topics to Be Covered in Today’s Lecture

- Section 4.8: Continuous Compounding
- Section 4.9: Interest Rates that vary over time
Section 4.8
Continuous Compounding

• Recall:
  – $EAIR = i = (1 + r/m)^m - 1$
  – What happens if we let $m$ approach infinity?
  – That means an infinite number of compounding periods within a year or,
  – The time between compounding approaches “0”.
  – We will see that a limiting value will be approached for a given value of “r”
4.8 Derivation of Continuous Compounding

- We can state, in general terms for the EAIR:

\[ i = \left(1 + \frac{r}{m}\right)^m - 1 \]

Now, examine the impact of letting “m” approach infinity.
4.8 Derivation of Continuous Compounding

- We re-define the EAIR general form as:

\[
(1 + \frac{r}{m})^m - 1 = \left[ \left( 1 + \frac{r}{m} \right)^{\frac{m}{r}} \right]^r - 1
\]

Note – the term in brackets has the exponent changed but all is still the same....
4.8 Derivation of Continuous Compounding

• There is a reason for the re-definition.
• From the calculus of limits there is an important limit that is quite useful.
• Specifically:

\[
\lim_{h \to \infty} \left(1 + \frac{1}{h}\right)^h = e = 2.71828
\]
4.8 Derivation of Continuous Compounding

- Substituting we can see:

\[
\lim_{{m \to \infty}} \left( 1 + \frac{r}{m} \right)^{\frac{m}{r}} = e,
\]
4.8 Derivation of Continuous Compounding

• So that:

\[ i = \lim_{m \to \infty} \left[ \left(1 + \frac{r}{m}\right)^{\frac{m}{r}} \right]^r - 1 = e^r - 1. \]

Summarizing……...
4.8 Derivation of Continuous Compounding

- The EAIR when interest is compounded continuously is then:

\[ \text{EAIR} = e^r - 1 \]

Where “\( r \)” is the nominal rate of interest compounded continuously.

This is the max. interest rate for any value of “\( r \)” compounded continuously.
4.8 Derivation of Continuous Compounding

• Example:

• What is the true, effective annual interest rate if the nominal rate is given as:
  – $r = 18\%$, compounded continuously
  – Or, $r = 18\%$ c.c.

Solve $e^{0.18} - 1 = 1.1972 - 1 = 19.72\%/year$

The 19.72% represents the MAXIMUM EAIR for 18% compounded anyway you choose!
4.8 Finding “r” from the EAIR/cont. compounding

• To find the equivalent nominal rate given the EAIR when interest is compounded continuously, apply:

\[ r = \ln(1 + i) \]
4.8 Example

• Given \( r = 18\% \) per year, cc, find:
  - A. the effective monthly rate
  - B. the effective annual rate

a. \( r/\text{month} = 0.18/12 = 1.5\%/\text{month} \)

  Effective monthly rate is \( e^{0.015} - 1 = 1.511\% \)

b. The effective annual interest rate is \( e^{0.18} - 1 = 19.72\% \) per year.
4.8 Example

- An investor requires an effective return of at least 15% per year.
- What is the minimum annual nominal rate that is acceptable if interest on his investment is compounded continuously?

To start: \( e^r - 1 = 0.15 \)

Solve for “\( r \)” ........
4.8 Example

- \( e^r - 1 = 0.15 \)
- \( e^r = 1.15 \)
- \( \ln(e^r) = \ln(1.15) \)
- \( r = \ln(1.15) = 0.1398 = 13.98\% \)

A rate of 13.98% per year, cc. generates the same as 15% true effective annual rate.
4.8 Final Thoughts

• When comparing different statements of interest rate one must always compute to true, effective annual rate (EAIR) for each quotation.

• Only EAIR’s can be compared!

• Various nominal rates cannot be compared unless each nominal rate is converted to its respective EAIR!
Section 4.9
Interest Rates that vary over time

• In practice – interest rates do not stay the same over time unless by contractual obligation.
• There can exist “variation” of interest rates over time – quite normal!
• If required, how do you handle that situation?
4.9 Interest Rates that vary over time

• Best illustrated by an example.

• Assume the following future profits:

\[
\begin{align*}
0 & : \$70,000 \\
1 & : \$70,000 \\
2 & : \$35,000 \\
3 & : \$25,000
\end{align*}
\]

\[
\begin{align*}
(\text{P/F,7}\%,1) & \\
(\text{P/F,7}\%,2) & \\
(\text{P/F,9}\%,3) & \\
(\text{P/F,10}\%,4)
\end{align*}
\]
4.9 Varying Rates: Present Worth

- To find the Present Worth:
  - Bring each cash flow amount back to the appropriate point in time at the interest rate according to:

\[ P = F_1(P/F,i_1,1) + F_2(P/F,i_1)(P/F,i_2) + \ldots \]

\[ + F_n(P/F,i_1)(P/F,i_2)(P/F,i_3)\ldots(P/F,i_n,1) \]

*This Process can get computationally involved!*
4.9 Period-by-Period Analysis

- **$P_0 =$**:
  1. $7000(P/F,7\%,1)$
  2. $7000(P/F,7\%,1)(P/F,7\%,1)$
  3. $35000(P/F,9\%,1)(P/F,7\%,1)^2$
  4. $25000(P/F,10\%,1)(P/F,9\%,1)(P/F,7\%,1)^2$

Equals: $172,816$ at $t = 0$...

Work backwards one period at a time until you get to “0”.
4.9 Varying Rates: Approximation

• An alternative approach that approximates the present value:

• Average the interest rates over the appropriate number of time periods.

• Example:
  – \( \frac{7\% + 7\% + 9\% + 10\%}{4} = 8.25\% \);
  – Work the problem with an 8.25% rate;
  – Merely an approximation.
4.9 Varying Rates: Single, Future Cash Flow

- Assume the following Cash Flow:

<table>
<thead>
<tr>
<th>Year</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$10,000</td>
</tr>
<tr>
<td>1</td>
<td>$10,000(1.08)</td>
</tr>
<tr>
<td>2</td>
<td>$10,000(1.08)(1.09)</td>
</tr>
<tr>
<td>3</td>
<td>$10,000(1.08)(1.09)(1.10)</td>
</tr>
<tr>
<td>4</td>
<td>$10,000(1.08)(1.09)(1.10)(1.11)</td>
</tr>
</tbody>
</table>

Objective: Find $P_0$ at the varying rates

$$P_0 = $10,000(P/F,8\%,1)(P/F,9\%,1)(P/F,10\%,1)(P/F,11\%,1)$$

$$= $10,000(0.9259)(0.9174)(0.9091)(0.9009)$$

$$= $10,000(0.6957) = $6,957$$
4.9 Varying Rates: Observations

- We seldom evaluate problem models with varying interest rates except in special cases.
- If required, best to build a spreadsheet model
- A cumbersome task to perform.
Chapter 4 Summary

• Many applications use and apply nominal and effective compounding

• Given a nominal rate – must get the interest rate to match the frequency of the payments.

• Apply the effective interest rate per payment period.
  – PP >= CP (adjusting the interest period to match the payment period)
  – PP < CP (Consider inter-period compounding or not?)

• When comparing varying interest rates, must calculate the EAIR in order to compare.
Chapter Summary – cont.

• All time value of money interest factors require use of an effective (true) periodic interest rate.

• The interest rate, $i$, and the payment or cash flow periods must have the same time unit.

• One may encounter varying interest rates and an exact answer requires a sequence of interest rates – cumbersome!
Assignments and Announcements

- Homework 3 due a week from today
- Assignments due at the beginning of next class:
  - Answer the three online quizzes on chapter 4 on the text’s website.
  - Read chapter 5.1, 5.2 and 5.3