Chapter 2
Factors: How Time and Interest Affect Money

Session 4-5-6
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Topics to Be Covered in Today’s Lecture

- Section 2: How Time and Interest Affect Money
  - Single-Payment Factors (F/P and P/F)
  - Interest Tables
  - Present Value/Future worth of an uneven payment series
  - Uniform-Series Present Worth Factor and Capital Recovery Factor P/A and A/P Factors
  - Sinking Fund Factor and Uniform-Series compound amount factor
Cash Flow Diagrams - Exercise

• An engineer wants to deposit an amount $P$ now such that she can withdraw an equal annual amount $2500$ per year for the first 3 years starting one year after the deposit and a different annual withdrawal of $3000$ per year for the following 5 years. How would the cash flow diagram appear if the interest rate were $7\%$ per year?
Single-payment Factors (F/P and P/F)

• **F/P**: To find $F$ given $P$
  - To find a certain future amount of money $F$ (compound amount), given the present amount of money $P$, a number of interest periods $n$, and an interest rate $i\%$
Single-payment Factors (F/P and P/F)

• **P/F**: To find *P* given *F*
  - To find the present worth *P*, given a future amount *F*, a number of interest periods *n*, and an interest rate *i%*
  - **P/F factor brings a single future sum back to a specific point in time.**
Single-payment Factors (F/P and P/F)

- \( F = P(1+i)^n \) and \( P = F \left( \frac{1}{(1+i)^n} \right) \)
- \((1+i)^n:\)
  - \( F/P \) factor,
  - Single Payment Compound Amount Factor (SPCAF),
  - \((F/P, i,n)\)
- \(1/(1+i)^n:\)
  - \( P/F \) factor,
  - Single Payment Present Worth Factor (SPPWF),
  - \((P/F, i,n)\)
Example- F/P Analysis

- Example: P= $1,000; n=3; i=10\%
- What is the future value, F?

\[
P = 1,000 \\
i = 10\% \text{/year}
\]

\[
F = ??
\]

\[
F_3 = 1,000[F/P, 10\%, 3] = 1,000[1.10]^3
\]

\[
= 1,000[1.3310] = 1,331.00
\]
Example – P/F Analysis

• Assume \( F = \$100,000 \), 9 years from now. What is the present worth of this amount now if \( i = 15\% \)?

\[
P_0 = \$100,000 \left( \frac{P}{F}, 15\%, 9 \right) = \$100,000 \left( \frac{1}{(1.15)^9} \right)
\]

\[
= \$100,000 \times 0.2843 = \$28,430 \text{ at time } t = 0
\]
Interest Tables

• Formulas can be used for calculating P or F, but interest tables can make some calculations easier
• Interest tables given at the back of the book
• Look up P/F (Present Worth), or F/P (Compound Amount) factors for a given interest rate and given number of interest periods
• When a given interest rate is not given at the back of the book, can use 1) Formulas (exact) 2) Interpolation (approximation)
Interest Tables Exercise

• If you had $2000 now and invested it at 10% how much would it be worth in 8 years?

• Suppose that $1000 is to be received in 5 years. At an annual interest rate of 12%, what is the present worth of this amount?
Present Value/Future worth of an uneven payment series

- What if payments are not uniform – some amount needs to be withdrawn 2 years from now, a different amount 5 years from now? How will you calculate future worth or present value in this case?
Present Value/Future worth of an uneven payment series – exercise

• Calculate the worth 5 years from now of an investment of $200 made 2 years from now and an investment of $100 made 4 years from now
Present Value/Future worth of an uneven payment series – exercise

• Wilson Technology a growing machine shop, wishes to set aside money now to invest over the next 4 years in automating its customer service department. The company can earn 10% on a lump sum deposited now and wishes to withdraw the money in the following increments
Present Value of an uneven payment series – exercise

• 1) Year 1: $25000 to purchase a computer and database software designed for customer service use
• 2) Year 2: $3000 to purchase additional hardware to accommodate the anticipated growth in use of the system
3) Year 3: No expenses
4) Year 4: $5000 to purchase software upgrades

How much money should the company set aside now?
Present Worth of an even payment series – exercise

• What is the present worth of a series of annual receipts of $100 that you will be receiving for three years starting next year?
Uniform Series Present Worth

- **Annuity Cash Flow**

$$P = ??$$

$A \text{ per period}$
Uniform Series Present Worth

• Desire an expression for the present worth – $P$ of a stream of equal, end of period cash flows - $A$

\[
P = ??
\]

$0 \quad 1 \quad 2 \quad 3 \quad \cdots \quad n-1 \quad n$

$A = \text{given}$
Uniform Series Present Worth

• Write a Present worth expression

\[
P = A \left[ \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n} \right] \tag{1}
\]

Term inside the brackets is a geometric progression.
Mult. This equation by 1/(1+i) to yield a second equation
Uniform Series Present Worth

• The second equation

\[
\frac{P}{1+i} = A \left[ \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \ldots + \frac{1}{(1+i)^n} + \frac{1}{(1+i)^{n+1}} \right] \]  \[2\]

To isolate an expression for \( P \) in terms of \( A \), subtract Eq [1] from Eq. [2]. Note that numerous terms will drop out.
Uniform Series Present Worth and Capital Recovery Factors

- Setting up the subtraction

\[
\frac{P}{(1+i)} = A \left[ \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4} + \ldots + \frac{1}{(1+i)^n} - \frac{1}{(1+i)^{n+1}} \right] \tag{2}
\]

\[
- P = A \left[ \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \ldots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n} \right] \tag{1}
\]

\[
= \frac{-i}{1+i} P = A \left[ \frac{1}{(1+i)^{n+1}} - \frac{1}{(1+i)} \right] \tag{3}
\]
Uniform Series Present Worth and Capital Recovery Factors

- Simplifying Eq. [3] further

\[
\frac{-i}{1+i} P = A \left[ \frac{1}{(1+i)^{n+1}} - \frac{1}{1+i} \right]
\]

\[
P = \frac{A}{-i} \left[ \frac{1}{(1+i)^{n+1}} - 1 \right] \rightarrow P = A \left[ \frac{(1+i)^n - 1}{i(1+i)^n} \right] \text{ for } i \neq 0
\]
Capital Recovery Factor (A/P)

- **A/P**: To find $A$ given $P$
  - Given an amount $P$ now, what is the annual $A$ worth over $n$ years when the interest rate is $i$.

$P$ is known
• \( P/A \) versus \( A/P \)

\[
P = A \left[ \frac{(1+i)^n - 1}{i(1+i)^n} \right], \quad i \neq 0
\]

\[
P/A \text{ factor} \quad (P/A,i,n)
\]

\[
\left[ \frac{(1+i)^n - 1}{i(1+i)^n} \right]
\]

\[
A/P \text{ factor} \quad (A/P,i,n)
\]

\[
A = P \left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right], \quad i \neq 0
\]

\[
\left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right]
\]

Uniform Series Present Worth Factor

Capital Recovery Factor
P/A example

• How much money should you be willing to pay now for a guaranteed $600 per year for 9 years starting next year, at a rate of return of 16% per year? (Answer using calculation and interest tables)
A/P Example

• Suppose you are entitled a payment of 30,000 a year for the next 9 years starting next year. What is the present worth of this annual payment to you? (Answer using calculation and interest tables)
More exercises

• BioGen, a small biotechnology firm, has borrowed $250,000 to purchase laboratory equipment for gene splicing. The loan carries an interest rate of 8% per year and is to be repaid in equal annual installments over the next 6 years. Compute the amount of this annual installment.
More exercises

• Suppose you are entitled a payment of 30,000 a year for the next 9 years starting next year from an investment firm. What is the present worth of this annual payment to you?
Sinking Fund Factor and Uniform-Series Compound Amount Factor

- Annuity Cash Flow

Find $A$ given the Future amt. - $F$
Sinking Fund and Series Compound amount factors (A/F and F/A)

• Take advantage of what we already have

• Recall:

\[
P = F \left[ 1 \right] 
\]

\[
A = P \left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right] 
\]

Substitute “P” and simplify!
F/A and A/F Derivations

- Annuity Cash Flow

$A per period

Find $F given the $A amounts
Example 2.6

- How much money must Carol deposit every year starting 1 year from now at 5.5% per year in order to accumulate $6000 seven years from now?

\[ A = ? \]

(a)

\[ i = 5\frac{1}{2}\% \]

\[ F = $6000 \]
Example 2.6

- Solution

- The cash flow diagram from Carol's perspective fits the A/F factor.

- \( A = 6000 \ (A/F, 5.5\%, 7) = 6000(0.12096) = \text{$725.76 per year} \)

- The A/F factor value of 0.12096 was computed using the A/F factor formula
Assignments

- Assignments due at the beginning of next class:
  - Read all the textbooks examples from sections 2.1, 2.2, 2.3
  - Read sections 2.4, 2.5, 2.6
- Hw1 due at the beginning of class one week from today.
Topics to Be Covered in Today’s Lecture

- Section 2: How Time and Interest Affect Money
  - Interpolation in interest tables
  - Arithmetic Gradient Factors
  - Geometric Gradient Factors
Arithmetic Gradient Factors

• In applications, the annuity cash flow pattern is not the only type of pattern encountered

• Two other types of end of period patterns are common
  • The Linear or arithmetic gradient
  • The geometric (% per period) gradient

• This section presents the Arithmetic Gradient
Arithmetic Gradient Factors

• An arithmetic (linear) Gradient is a cash flow series that either increases or decreases by a constant amount over n time periods.

• A linear gradient is always comprised of TWO components:
  • The Gradient component
  • The base annuity component

• We need an expression for the Present Worth of an arithmetic gradient
Linear Gradient Example

• Assume the following:

This represents a positive, increasing arithmetic gradient
Example: Linear Gradient

• **Typical Negative, Increasing Gradient:**
  \[ G = $50 \]
  
  The Base Annuity = $1500

\[ \$1500 \quad \$1550 \quad \$1600 \quad \$1650 \quad \$1500 \quad +(n - 2)50 \quad \$1500 \quad +(n - 1)50 \]
Example: Linear Gradient

- Desire to find the Present Worth of this cash flow

\[
\text{The Base Annuity} = \$1500
\]
Arithmetic Gradient Factors

• The “G” amount is the constant arithmetic change from one time period to the next.

• The “G” amount may be positive or negative!

• The present worth point is always one time period to the left of the first cash flow in the series or, 

• Two periods to the left of the first gradient cash flow!
Derivation: Gradient Component Only

- Focus Only on the gradient Component

"0" G

Removed Base annuity
Present Worth Point…

The Present Worth Point of the Gradient
Gradient Component

The Gradient Component

The Present Worth Point of the Gradient
Present Worth Point…

• **PW of the Base Annuity is at t = 0**

\[ \text{PW of the Base Annuity is at } t = 0 \]

**The Present Worth Point of the Gradient**

• **PW BASE Annuity** = $100\left(\frac{P}{A, i\%}{,7}\right)$
Present Worth: Linear Gradient

• The present worth of a linear gradient is the present worth of the two components:
  – 1. The Present Worth of the Gradient Component and,
  – 2. The Present Worth of the Base Annuity flow
  – Requires 2 separate calculations!
Present Worth: Gradient Component

• The PW of the Base Annuity is simply the Base Annuity \(-A(P/A, i\%, n)\) factor
• What is needed is a present worth expression for the gradient component cash flow.
• We need to derive a closed form expression for the gradient component!
Present Worth: Gradient Component

• General CF Diagram – Gradient Part Only

We want the PW at time $t = 0$ (2 periods to the left of $1G$)
To Begin- Derivation of $(P/G, i\%, n)$

\[ P = G(P/F, i\%, 2) + 2G(P/F, i\%, 3) + \ldots \]
\[ [(n-2)G](P/F, i\%, n-1) + [(n-1)G](P/F, i\%, n) \]

Next Step:

Factor out $G$ and re-write as .....
Factoring G out…. P/G factor

\[ P = G \{ (P/F,i\%,2) + 2(P/F,i\%,3) + \ldots + (n-1)(P/F,i\%,n) \} \]

What is inside of the \{ \}’s?
Replace (P/F’s) with closed-form

\[ P = G \left[ \frac{1}{(1+i)^2} + \frac{2}{(1+i)^3} + \ldots + \frac{n-2}{(1+i)^{n-1}} + \frac{n-1}{(1+i)^n} \right] \] [1]

Multiply both sides by \((1+i)\)
Mult. Both Sides By \((n+1)\)…..

\[
P(1+i)^1 = G \left[ \frac{1}{(1+i)^1} + \frac{2}{(1+i)^2} + \ldots + \frac{n-2}{(1+i)^{n-2}} + \frac{n-1}{(1+i)^{n-1}} \right] [2]
\]

• We have 2 equations [1] and [2].
• Next, subtract [1] from [2] and work with the resultant equation.
Subtracting [1] from [2]…..

\[ P(1+i)^1 = G \left[ \frac{1}{(1+i)^1} + \frac{2}{(1+i)^2} + \ldots + \frac{n-2}{(1+i)^{n-2}} + \frac{n-1}{(1+i)^{n-1}} \right] \]

\[ - \quad P = G \left[ \frac{1}{(1+i)^2} + \frac{2}{(1+i)^3} + \ldots + \frac{n-2}{(1+i)^{n-1}} + \frac{n-1}{(1+i)^n} \right] \]

\[ P = \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n} \right] \]
The P/G factor for i and N

\[ P = \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n} \right] \]

\((P / G, i\% , N)\) factor
Further Simplification on P/G

\[
(P / G, i\%, N) = \frac{(1 + i)^N - iN - 1}{i^2 (1 + i)^N}
\]

Remember, the present worth point of any linear gradient is 2 periods to the left of the 1-G cash flow or, 1 period to the left of the “0-G” cash flow.

\[
P = G(P / G, i, n)
\]
Gradient Example

• Consider the following cash flow

Find the present worth if \( i = 10\% / \text{yr}; \, n = 5 \text{ yrs} \)
Gradient Example- Base Annuity

- First, The Base Annuity of $100/period
  \[ A = +$100 \]

  \[ \text{PW}(10\%) \text{ of the base annuity} = $100(P/A,10\%,5) \]
  \[ \text{PW}_{\text{Base}} = $100(3.7908) = $379.08 \]

  Not Finished: We need the PW of the gradient component and then add that value to the $379.08 amount
Focus on the Gradient Component

We desire the PW of the Gradient Component at $t = 0$

$$P_{G@t=0} = G\left( \frac{P}{G},10\%,5 \right) = $100\left( \frac{P}{G},10\%,5 \right)$$
The Set Up

\[ P_{G@t=0} = G\left(\frac{P}{G}, 10\%, 5\right) = $100\left(\frac{P}{G}, 10\%, 5\right) \]

\[
P = \frac{G}{i} \left[ \frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N} \right]
\]

Could substitute \( n=5 \), \( i=10\% \) and \( G = $100 \) into the \( P/G \) closed form to get the value of the factor.
PW of the Gradient Component

\[ P_{G@t=0} = G(P/G,10\%,5) = $100(P/G,10\%,5) \]

\[ P/G,10\%,5) \]

Sub. G=$100;i=0.10;n=5

\[ \frac{G}{i} \left[ \frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N} \right] \]

\[ 6.8618 \]

Calculating or looking up the \((P/G,10\%,5)\) factor yields the following:

\[ P_{t=0} = $100(6.8618) = $686.18 \] for the gradient PW
Gradient Example: Final Result

- $\text{PW}(10\%)_{\text{Base Annuity}} = 379.08$
- $\text{PW}(10\%)_{\text{Gradient Component}} = 686.18$
- Total $\text{PW}(10\%) = 379.08 + 686.18$
- Equals $1065.26$

- Note: The two sums occur at $t = 0$ and can be added together – concept of equivalence
Example Summarized

This Cash Flow...

Is equivalent to $1065.26 at time 0 if the interest rate is 10% per year!
Shifted Gradient Example: $i = 10\%$

- Consider the following Cash Flow

1. This is a “shifted” negative, decreasing gradient.
2. The PW point in time is at $t = 3$ (not $t = 0$)
Shifted Gradient Example

Consider the following Cash Flow

- The PW @ t = 0 requires getting the PW @ t = 3;
- Then using the P/F factor move PW₃ back to t = 0
Shifted Gradient Example

- Consider the following Cash Flow

0           1             2             3            4          5             6             7
$600
$550
$500
$450
$600
$550
$500
$450

- The base annuity is a $600 cash flow for 3 time periods
Shifted Gradient Example: Base Annuity

• **PW of the Base Annuity: 2 Steps**

\[ P_0 = P_3 \left( \frac{P}{F}, 10\%, 3 \right) \]

\[ P_3 = -600 \left( \frac{P}{A}, 10\%, 4 \right) \]

\[ P_0 = \left[ -600 \left( \frac{P}{A}, 10\%, 4 \right) \right] \left( \frac{P}{F}, 10\%, 3 \right) \]

\[ P_{0\text{-base annuity}} = \boxed{-1428.93} \]
Shifted Gradient Example: Gradient

- PW of Gradient Component: $G = -$50

\[ P_0 = P_3(P/F, 10\%, 3) \]

\[ P_0 - \text{grad} = \{ +50(P/G, 10\%, 4) \}(P/F, 10\%, 3) = -$164.46 \]
Extension – The A/G factor

- A/G converts a linear gradient to an equivalent annuity cash flow.
- Remember, at this point one is only working with gradient component
- There still remains the annuity component that you must also handle separately!
The A/G Factor

- Convert G to an equivalent A

\[ A = G\left(\frac{P}{G}, i, n\right)\left(\frac{A}{P}, i, n\right) \]

How to do it..........
A/G factor using A/P with P/G

\[
P = \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n} \right] \left[ \frac{i(1+i)^n}{(1+i)^n - 1} \right]
\]

\[(A/P, i, n)\]

The results follow…. 
Resultant A/G factor

\[
P = \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n} \right]
\]

\[
(A/P, i, n)
\]

\[
(A/G, i, n) = A = G \left[ \frac{1}{i} - \frac{n}{(1+i)^n - 1} \right]
\]
Terminology

• Let $P_A$ be the present worth of the base annual amount $A_A$ (which is a series of equal end-of-period cash flows).

• Let $P_G$ be the present worth of the gradient cash flow.

• Let $P_T$ be the total present worth,
  - $P_T = P_A + P_G$ if the gradient is positive
  - $P_T = P_A - P_G$ if the gradient is negative

• Let $A_T$ be the total annual worth for $n$ years of the series of gradient cash flows including the base annual amount $A_A$. 
\[ P_G = G \left\{ \frac{1}{i} \left[ \frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n} \right] \right\} \]

\( (P/G,i,n) \) factor

\( P/G \) factor

Arithmetic Gradient

Present Worth Factor
Arithmetic Gradients

- \( P_T = P_A + P_G \) for positive gradients and
- \( P_T = P_A - P_G \) for negative gradients
- So, how would you find \( P_A \)?
- Given that you have the \( P_T \) how would you find \( A_T \)?
Assignments

- Assignments due at the beginning of next class:
  - Read all the textbooks examples from sections 2.4, 2.5
  - Read sections 2.6, 2.7, 2.8
Topics to Be Covered in Today’s Lecture

- Section 2: How Time and Interest Affect Money
  - The A/G factor
  - Geometric Gradient Factors
  - Determination of an unknown interest rate
  - Determination of an unknown number of periods
Extension – The A/G factor

• A/G converts a linear gradient to an equivalent annuity cash flow.
• Remember, at this point one is only working with gradient component
• There still remains the annuity component that you must also handle separately!
The A/G Factor

• Convert G to an equivalent A

\[ A = G(P/G, i, n)(A/P, i, n) \]

How to do it............
A/G factor using A/P with P/G

\[
A = \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n - 1} \right]
\]

\((A/P,i,n)\)

The results follow.....
Resultant A/G factor

\[ A = \frac{G}{i} \left[ \frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n} \right] - \frac{i(1+i)^n}{(1+i)^n - 1} \]
Terminology

- Let $P_A$ be the present worth of the base annual amount $A_A$ (which is a series of equal end-of-period cash flows).
- Let $P_G$ be the present worth of the arithmetic gradient cash flow.
- Let $P_T$ be the total present worth,
  - $P_T = P_A + P_G$ if the gradient is positive
  - $P_T = P_A - P_G$ if the gradient is negative
- Let $A_T$ be the total annual worth for $n$ years of the series of gradient cash flows including the base annual amount $A_A$. 
\[ P_G = G \left\{ \frac{1}{i} \left[ \frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n} \right] \right\} \]

\[ \left\{ \frac{1}{i} \left[ \frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n} \right] \right\} \]

(P/G, i, n) factor

P/G factor

Arithmetic Gradient Present Worth Factor
Arithmetic Gradients

- $P_T = P_A + P_G$ for positive gradients and
- $P_T = P_A - P_G$ for negative gradients
- So, how would you find $P_A$?
- Given that you have the $P_T$ how would you find $A_T$?
Geometric Gradients

• An arithmetic (linear) gradient changes by a fixed dollar amount each time period.

• A GEOMETRIC gradient changes by a fixed percentage each time period.

• We define a UNIFORM RATE OF CHANGE (%) for each time period.

• Define “$g$” as the constant rate of change in decimal form by which amounts increase or decrease from one period to the next.
Geometric Gradients: Increasing

- Typical Geometric Gradient Profile
- Let $A_1$ = the first cash flow in the series
Geometric Gradients: Decreasing

- Typical Geometric Gradient Profile
- Let $A_1$ = the first cash flow in the series
Geometric Gradients: Derivation

• First Major Point to Remember:
  • $A_1$ does NOT define a Base Annuity;
  • There is no BASE ANNUITY for a Geometric Gradient!

• The objective is to determine the Present Worth one period to the left of the $A_1$ cash flow point in time

• Remember: The PW point in time is one period to the left of the first cash flow – $A_1$!
Geometric Gradients: Derivation

• For a Geometric Gradient the following parameters are required:
  • The interest rate per period – \( i \)
  • The constant rate of change – \( g \)
  • No. of time periods – \( n \)
  • The starting cash flow – \( A_1 \)
Geometric Gradients: Starting

- $P_g$ = The $A_j$’s time the respective $(P/F,i,j)$ factor

- Write a general present worth relationship to find $P_g$...

$$P_g = \frac{A_1}{(1+i)^1} + \frac{A_1(1+g)}{(1+i)^2} + \frac{A_1(1+g)^2}{(1+i)^3} + ... + \frac{A_1(1+g)^{n-1}}{(1+i)^n}$$

Now, factor out the $A_1$ value and rewrite as..
Geometric Gradients

\[ P_g = A_l \left[ \frac{1}{(1+i)} + \frac{(1+g)^1}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \ldots + \frac{(1+g)^{n-1}}{(1+i)^n} \right] \] (1)

Multiply both sides by \( \frac{(1+g)}{(1+i)} \) to create another equation

\[ P_g \frac{(1+g)}{(1+i)} = A_l \frac{(1+g)}{(1+i)} \left[ \frac{1}{(1+i)} + \frac{(1+g)^1}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \ldots + \frac{(1+g)^{n-1}}{(1+i)^n} \right] \] (2)

Subtract (1) from (2) and the result is.....
Geometric Gradients

\[ P_g \left( \frac{1+g}{1+i} - 1 \right) = A_1 \left[ \frac{(1 + g)^n}{(1 + i)^{n+1}} - \frac{1}{1 + i} \right] \]

Solve for \( P_g \) and simplify to yield….

\[ P_g = A_1 \left[ \frac{1 - \left( \frac{1+g}{1+i} \right)^n}{i - g} \right] \quad g \neq i \]
Geometric Gradient P/A factor

\[ P_g = A_1 \left[ 1 - \left( \frac{1+g}{1+i} \right)^n \right] \frac{1-i}{i-g} \quad g \neq i \]

• This is the (P/A, \(g, i, n\)) factor and is valid if \(g\) is not equal to \(i\).
Geometric Gradient P/A factor

• Note: If \( g = i \) we have a division by “0” – undefined.

• For \( g = i \) we can derive the closed form PW factor for this special case.

• We substitute \( i \) for \( g \) into the \( P_g \) relationship to yield:
Geometric Gradient: $i = g$ Case

$$P_g = A_1 \left( \frac{1}{(1+i)} + \frac{1}{(1+i)} + \frac{1}{(1+i)} + \ldots + \frac{1}{(1+i)} \right)$$

$$P_g = \frac{nA_1}{(1+i)}$$

For the case $i = g$
Geometric Gradients: Summary

\[ P_g = A_1 \left( \frac{1 - (1 + g)^n}{1 + i} \right) \]

- \( g \neq i \)

**Case:** \( g = i \)

\[ P_g = \frac{nA_1}{(1 + i)} \]
Geometric Gradient: Notes

• The geometric gradient requires knowledge of:

• $A_1$, i, n, and g

• There exist an infinite number of combinations for i, n, and g: Hence one will not find tabulated tables for the $(P/A, g,i,n)$ factor.
Geometric Gradient: Notes

• You have to calculated either from the closed form for each problem or apply a pre-programmed spreadsheet model to find the needed factor value

• No spreadsheet built-in function for this factor!
Geometric Gradient: Example

- Assume maintenance costs for a particular activity will be $1700 one year from now.

- Assume an annual increase of 11% per year over a 6-year time period.
Geometric Gradient: Example

• If the interest rate is 8% per year, determine the present worth of the future expenses at time $t = 0$.

• First, draw a cash flow diagram to represent the model.
Geometric Gradient Example (+g)

- \( g = +11\% \text{ per period}; \ A_1 = \$1700; \ i = 8\%/\text{yr} \)

\[ \begin{align*}
0 & \quad \$1700 \\
1 & \quad \$1700(1.11) \\
2 & \quad \$1700(1.11)^2 \\
3 & \quad \$1700(1.11)^3 \\
4 & \quad \$1700(1.11)^4 \\
5 & \quad \$1700(1.11)^5 \\
6 & \quad \$1700(1.11)^6 \\
7 & \quad \$1700(1.11)^7 \\
\end{align*} \]

\[ \text{PW}(8\%) = ?? \]
Solution

- $P = 1700(P/A, 11\%, 8\%, 7)$
- Need to calculate the $P/A$ factor from the closed-form expression for a geometric gradient.

$P_g = A_1 \left[ \frac{1 - \left( \frac{1+g}{1+i} \right)^n}{i-g} \right] \quad g \neq i$

$11980.44$
Geometric Gradient ( -g )

- Consider the following problem with a negative growth rate – $g$.

\[ A_1 = $1000 \]

\[ \begin{array}{c}
0 & 1 & 2 & 3 & 4 \\
$1000 & $900 & $810 & $729 \\
\end{array} \]

\[ g = -10\% / yr; \ i = 8\%; \ n = 4 \]

We simply apply a “g” value = -0.10
Geometric Gradient (-g value)

• Evaluate:

For a negative g value = -0.10

\[ P_g = A_1 \left[ \frac{1 - \left( \frac{1+g}{1+i} \right)^n}{i-g} \right] \quad g \neq i \]

\[ \text{=} \$2876.37 \]
Determination of an unknown interest rate and number of periods

• Interest Rate Unknown
  – Given: \( P, F, n \)
  – Determine \( i \)

• Number of periods unknown
  – Given: \( P, F, i \)
  – Determine \( n \)

• Use formulas to calculate
Determination of an unknown interest rate exercise

• If Laurel can make an investment in a friend’s business of $3000 now in order to receive $5000 five years from now, determine the rate of return. If Laurel can receive 7% per year interest on a certificate of deposit, which investment should be made?
Determination of an unknown number of interest periods exercise

• How long will it take for $1000 to double if the interest rate is 5% per year? Calculate the exact value. How different are the estimates if the rate is 25%?
Assignments and Announcements

- Assignments due at the beginning of next class:
  - Read all the textbook examples from sections 2.6, 2.7, 2.8
  - Read chapter 3
  - We have a quiz next time